Bootstrapping pre-averaged realized volatility under market microstructure noise *

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August 29, 2013

Abstract

The main contribution of this paper is to propose a bootstrap method for inference on integrated volatility based on the pre-averaging approach of Jacod et al. (2009), where the pre-averaging is done over all possible overlapping blocks of consecutive observations. The overlapping nature of the pre-averaged returns implies that these are k_n-dependent with k_n growing slowly with the sample size n. This motivates the application of a blockwise bootstrap method. We show that the "blocks of blocks" bootstrap method suggested by Politis and Romano (1992) (and further studied by Bühlmann and Künsch (1995)) is valid only when volatility is constant. The failure of the blocks of blocks bootstrap is due to the heterogeneity of the squared pre-averaged returns when volatility is stochastic. To preserve both the dependence and the heterogeneity of squared pre-averaged returns, we propose a novel procedure that combines the wild bootstrap with the blocks of blocks bootstrap. We provide a proof of the first order asymptotic validity of this method for percentile intervals. Our Monte Carlo simulations show that the wild blocks of blocks bootstrap improves the finite sample properties of the existing first order asymptotic theory. We use empirical work to illustrate its use in practice.

Keywords: High frequency data, realized volatility, pre-averaging, market microstructure noise, wild bootstrap, block bootstrap.

*We would like to thank Ilze Kalnina, Kevin Sheppard and Neil Shephard for many useful comments and discussions. This work was supported by grants FQRSC-ANR and SSHRC. In addition, Ulrich Hounyo acknowledges support from CREATES - Center for Research in Econometric Analysis of Time Series (DNRF78), funded by the Danish National Research Foundation, as well as support from the Oxford-Man Institute of Quantitative Finance. Finally, Nour Meddahi benefited from the financial support of the chair "Marché des risques et création de valeur" Fondation du risque/SCOR.

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1 Introduction

Estimation of integrated volatility is complicated by the existence of market microstructure noise. This noise represents the discrepancy between the true efficient price of an asset and its observed counterpart and is caused by a multitude of market microstructure effects (such as bid-ask bounds, the discreteness of price changes and the existence of rounding errors, the gradual response of prices to a block trade, the existence of data recording errors such as prices entered as zero, misplaced decimal points, etc).

Realized volatility, computed as the sum of squared intraday returns, is not consistent for integrated volatility under the presence of market microstructure noise. This has motivated the development of alternative estimators. One popular method is the pre-averaging approach first introduced by Podolskij and Vetter (2009) and further studied by Jacod et al. (2009). The basic underlying idea consists of first averaging out the noise by computing pre-averaged returns and then computing a realized volatility-like estimator using the pre-averaged returns. Although the pre-averaged realized volatility estimator is consistent for integrated volatility, its convergence rate is much slower than that of realized volatility and this can result in finite sample distortions that persist even at very large sample sizes. For this reason, the bootstrap is a useful alternative method of inference in this context.

In this paper, we propose a bootstrap method that can be used to estimate the distribution and the variance of the pre-averaged realized volatility estimator of Jacod et al. (2009). Our proposal is to resample the pre-averaged returns instead of resampling the original noisy returns. To be valid, the bootstrap needs to mimic the dependence and heterogeneity properties of the (squared) pre-averaged returns. When pre-averaging occurs over overlapping blocks of returns, as in Jacod et al. (2009), the squared pre-averaged returns are k_n-dependent, where k_n denotes the block length of the interval over which the pre-averaging is done and n denotes the sample size. Since k_n is proportional to $\sqrt{n}$, $k_n \to \infty$ as $n \to \infty$, which implies that the pre-averaged returns are strongly dependent. This suggests that a block bootstrap applied to the pre-averaged returns is appropriate and its application amounts to a "blocks of blocks" bootstrap, as proposed by Politis and Romano (1992) and further studied by Bühlmann and Künsch (1995) (see also Künsch (1989)). Nevertheless, as we show here, such a bootstrap scheme is only consistent in our setup when volatility is constant. The reason is that squared pre-averaged returns are heterogenously distributed (in particular, their mean and variance are time-varying) and this creates a bias term in the blocks of blocks bootstrap variance estimator when volatility is stochastic. Thus, to handle both the dependence and heterogeneity of the squared pre-averaged returns, we propose a novel bootstrap approach that combines the wild bootstrap with the blocks of blocks bootstrap. We name this novel approach the wild blocks of blocks bootstrap. Our main contribution is to show that this method consistently estimates the variance and the entire distribution of the pre-averaged estimator of Jacod et al. (2009).

The pre-averaging approach can also be implemented with non-overlapping intervals, as in Podolskij and Vetter (2009). Gonçalves, Hounyo and Meddahi (2013) study the consistency of the wild bootstrap for this estimator. The wild bootstrap exploits the asymptotic independence of the pre-averaged returns when these are computed over non-overlapping intervals. This method is no longer valid when overlapping intervals are used to compute pre-averaged returns since these are strongly dependent. For this reason, a new bootstrap method is needed for the Jacod et al.'s (2009) approach. Although the wild blocks of blocks bootstrap that we propose here requires the choice of an additional tuning parameter (the block size), we suggest an empirical procedure to select the block size that performs well in our simulations.

Other estimators of integrated volatility that are consistent under market microstructure noise include the subsampling approach of Zhang et al. (2005) and the realized kernel estimator of Barndorff-Nielsen et al. (2008) (the maximum likelihood-based estimator of Xiu (2010) is also a recent addition to this literature). The bootstrap could also be useful for inference in the context of these estimators. Indeed, Zhang et al. (2011) showed that the asymptotic normal approximation is often inaccurate for the subsampling realized volatility estimator, whose finite sample distribution is skewed and heavy tailed. They proposed Edgeworth corrections for this estimator as a way to improve upon the standard normal approximation. Similarly, Bandi and Russell (2011) discussed the limitations of asymptotic approximations in the context of realized kernels and proposed an alternative solution. The main reason why we focus on the pre-averaging approach here is that it naturally lends itself to the bootstrap. In particular, we resample the pre-averaged returns instead of the individual returns and exploit the dependence and heterogeneity properties of the pre-averaged returns to prove the consistency of the bootstrap. In addition, the pre-averaging approach has some important advantages compared to the preceding methods, for example it can easily estimate the integrated quarticity or other functionals of volatility.

The rest of this paper is organized as follows. In the next section, we first introduce the setup, our assumptions and review the existing asymptotic theory of Jacod et al. (2009). Section 3 contains the bootstrap results. In Section 3.1 we show that the blocks of blocks bootstrap is consistent only when volatility is constant whereas Section 3.2 describes the wild blocks of blocks bootstrap and shows its consistency under stochastic volatility and i.i.d. noise. Section 4 presents the simulation results whereas Section 5 contains an empirical application. Section 6 concludes. Two appendices are provided. Appendix A contains the tables with simulation results whereas Appendix B is a mathematical appendix with the proofs.

A word on notation. In this paper, and as usual in the bootstrap literature, P^* (E^* and Var^*) denotes the probability measure (expected value and variance) induced by the bootstrap resampling, conditional on a realization of the original time series. In addition, for a sequence of bootstrap statistics Z_n^*, we write $Z_n^* = o_{P^*}(1)$ in probability, or $Z_n^* \to^{P^*} 0$, as $n \to \infty$, in probability, if for any $\varepsilon > 0$, $\delta > 0$, $\lim_{n\to\infty} P\left[P^*\left(|Z_n^*| > \delta\right) > \varepsilon\right] = 0$. Similarly, we write $Z_n^* = O_{P^*}(1)$ as $n \to \infty$, in probability if for all $\varepsilon > 0$ there exists a $M_\varepsilon < \infty$ such that $\lim_{n\to\infty} P\left[P^*\left(|Z_n^*| > M_\varepsilon\right) > \varepsilon\right] = 0$. Finally, we write $Z_n^* \to^{d^*} Z$ as $n \to \infty$, in probability, if conditional on the sample, Z_n^* weakly converges to Z under P^*, for all samples contained in a set with probability P converging to one.

2 Setup, assumptions and review of existing results

2.1 Setup and assumptions

Let X denote the latent efficient log-price process defined on a probability space $(\Omega^0, \mathcal{F}^0, P^0)$ equipped with a filtration $(\mathcal{F}_t^0)_{t\geq 0}$. We model X as a Brownian semimartingale process defined by the equation

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s,\ t \geq 0, \tag{1}$$

where $a = (a_t)_{t\geq 0}$ is an adapted càdlàg drift process, $\sigma = (\sigma_t)_{t\geq 0}$ is an adapted càdlàg volatility process and $W = (W_t)_{t\geq 0}$ a standard Brownian motion.

The object of interest is the quadratic variation of X, i.e. the process

$$C_t = \int_0^t \sigma_s^2 ds,$$

also known as the integrated volatility. Without loss of generality, we let $t = 1$ and define $C_1 = \int_0^1 \sigma_s^2 ds$ as the integrated volatility of X over a given time interval $[0, 1]$, which we think of as a given day.

The presence of market frictions such as price discreteness, rounding errors, bid-ask spreads, gradual response of prices to block trades, etc, prevent us from observing the true efficient price process X. Instead, we observe a noisy price process Y, observed at time points $t = \frac{i}{n}$ for $i = 0, \ldots, n$, given by

$$Y_t = X_t + \epsilon_t,$$

where ϵ_t represents the noise term that collects all the market microstructure effects.

In order to make both X and Y measurable with respect to the filtration, we define a new probability space $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, P\right)$, which accommodates both processes. To this end, we follow Jacod et al. (2009) and assume one has a second space $\left(\Omega^1, (\mathcal{F}_t^1)_{t\geq 0}, P^1\right)$, where Ω^1 denotes $\mathbb{R}^{[0,1]}$ and $\mathcal{F}^1$ the product Borel-σ-field on Ω^1. Next, let Q_t be a probability measure on $\mathbb{R}$ (Q_t is the marginal distribution of ϵ_t). P^1 denotes the product measure $\otimes_{t\in[0,1]} Q_t$. The filtered probability space $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, P\right)$ on which the process Y lives is then defined with $\Omega = \Omega^0 \times \Omega^1$, $\mathcal{F} = \mathcal{F}^0 \times \mathcal{F}^1$, $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s^0 \times \mathcal{F}_s^1$, and $P = P^0 \otimes P^1$.

We assume that ϵ_t is centered and independent, conditionally on the efficient price process X. In addition, we assume that the conditional variance of ϵ_t is càdlàg. Assumption 1 below collects these assumptions.

Assumption 1.

(i) $E(\epsilon_t|X) = 0$ and ϵ_t and ϵ_s are independent for all $t \neq s$, conditionally on X.

(ii) $\alpha_t = E(\epsilon_t^2|X)$ is càdlàg and $E(\epsilon_t^8) < \infty$.

Assumption 1 amounts to Assumption (K) in Jacod et al. (2009). As they explain, this assumption is rather general, allowing for time varying variances of the noise and dependence between X and ϵ. See Jacod et al. (2009) for particular examples of market microstructure noise that satisfy Assumption 1.

2.2 The pre-averaged estimator and its asymptotic theory

We observe Y at regular time points $\frac{i}{n}$, for $i = 0, \ldots, n$, from which we compute n intraday returns at frequency $\frac{1}{n}$,

$$r_i \equiv Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}, \quad i = 1, \ldots, n.$$

Given that $Y = X + \epsilon$, we can write

$$r_i = \left(X_{\frac{i}{n}} - X_{\frac{i-1}{n}}\right) + \left(\epsilon_{\frac{i}{n}} - \epsilon_{\frac{i-1}{n}}\right) \equiv r_i^e + \Delta\epsilon_i,$$

where $r_i^e = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}$ denotes the $\frac{1}{n}$-frequency return on the efficient price process. Under Assumption 1, the order of magnitude of $\Delta\epsilon_i \equiv \epsilon_{\frac{i}{n}} - \epsilon_{\frac{i-1}{n}}$ is $O_P(1)$. In contrast, r_i^e is (conditionally on the path of σ and a) independent and heteroskedastic with (conditional) variance given by $\int_{(i-1)/n}^{i/n} \sigma_s^2 ds$. Thus, its order of magnitude is $O_P\left(n^{-1/2}\right)$. This decomposition shows that the noise completely dominates the observed return process as $n \to \infty$, implying that the usual realized volatility estimator is biased and inconsistent. See Zhang et al. (2005) and Bandi and Russell (2008).

To describe the Jacod et al. (2009) pre-averaging approach, let k_n be a sequence of integers which will denote the window length over which the pre-averaging of returns is done. Similarly, let g be a weighting function on $[0, 1]$ such that $g(0) = g(1) = 0$ and $\int_0^1 g(s)^2 ds > 0$, and assume g is continuous and piecewise continuously differentiable with a piecewise Lipschitz derivative g'. An example of a function that satisfies these restrictions is $g(x) = \min(x, 1-x)$.

We introduce the following additional notation. Let

$$\phi_1(s) = \int_s^1 g'(u)\, g'(u-s)\, du \quad \text{and} \quad \phi_2(s) = \int_s^1 g(u)\, g(u-s)\, du,$$

and for $i = 1, 2$, let $\psi_i = \phi_i(0)$. For instance, for $g(x) = \min(x, 1-x)$, we have that $\psi_1 = 1$ and $\psi_2 = 1/12$.

For $i = 0, \ldots, n-k_n+1$, the pre-averaged returns $\bar{Y}_i$ are obtained by computing the weighted sum of all consecutive $\frac{1}{n}$-horizon returns over each block of size k_n,

$$\bar{Y}_i = \sum_{j=1}^{k_n} g\left(\frac{j}{k_n}\right) r_{i+j}.$$

The effect of pre-averaging is to reduce the impact of the noise in the pre-averaged return. Specifically, as shown by Vetter (2008),

$$\bar{X}_i = \sum_{j=1}^{k_n} g\left(\frac{j}{k_n}\right)\left(X_{\frac{i+j}{n}} - X_{\frac{i+j-1}{n}}\right) = O_P\left(\sqrt{\frac{k_n}{n}}\right),$$

and

$$\bar{\epsilon}_i = \sum_{j=1}^{k_n} g\left(\frac{j}{k_n}\right)\left(\epsilon_{\frac{i+j}{n}} - \epsilon_{\frac{i+j-1}{n}}\right) = O_P\left(\frac{1}{\sqrt{k_n}}\right).$$

Thus, the impact of the noise is reduced the larger k_n is. To get the efficient $n^{-1/4}$ rate of convergence, Jacod et al. (2009) propose to choose a sequence of integers k_n such that the following assumption holds.

Assumption 2. For $\theta \in (0, \infty)$, we have that

$$\frac{k_n}{\sqrt{n}} = \theta + o\left(n^{-1/4}\right). \tag{2}$$

This choice implies that the orders of the two terms ($\bar{X}_i$ and $\bar{\epsilon}_i$) are balanced and equal to $O_P\left(n^{-1/4}\right)$. An example that satisfies (2) is $k_n = [\theta\sqrt{n}]$.

Based on the pre-averaged returns $\bar{Y}_i$, Jacod et al. (2009) propose the following estimator of integrated volatility,

$$PRV_n = \frac{1}{\psi_2 k_n}\sum_{i=0}^{n-k_n+1}\bar{Y}_i^2 - \frac{\psi_1}{2n\theta^2\psi_2}\sum_{i=1}^{n} r_i^2, \tag{3}$$

where ψ_1 and ψ_2 are as defined above.

The first term in (3) is an average of realized volatility-like estimators based on pre-averaged returns of length k_n whereas the second term is a bias correction term. As discussed in Jacod et al. (2009), this bias term does not contribute to the asymptotic variance of PRV_n.

In order to give the central limit theorem for PRV_n, we introduce the following numbers that are associated with g,

$$\Phi_{ij} = \int_0^1 \phi_i(s)\,\phi_j(s)\,ds, \quad \text{and} \quad \Psi_{ij} = -\int_0^1 s\phi_i(s)\,\phi_j(s)\,ds.$$

For the simple function $g(x) = \min(x, 1-x)$, $\Phi_{11} = 1/6$, $\Phi_{12} = 1/96$ and $\Phi_{22} = 151/80640$. Under Assumption 1 and (k_n, θ) satisfying (2), Jacod et al. (2009) show that as $n \to \infty$,

$$\frac{n^{1/4}\left(PRV_n - \int_0^1 \sigma_s^2 ds\right)}{\sqrt{V}} \to^{st} N(0,1), \tag{4}$$

where $\to^{st}$ denotes stable convergence, and

$$V = \frac{4}{\psi_2^2}\int_0^1 \left(\Phi_{22}\theta\sigma_s^4 + 2\Phi_{12}\frac{\sigma_s^2\alpha_s}{\theta} + \Phi_{11}\frac{\alpha_s^2}{\theta^3}\right) ds$$

is the conditional variance of PRV_n. To estimate V consistently, Jacod et al. (2009) propose

$$\begin{aligned} \hat{V}_n &= \frac{4\Phi_{22}}{3\theta\psi_2^4}\sum_{i=0}^{n-k_n+1}\overline{Y}_i^4 + \frac{4}{n\theta^3}\left(\frac{\Phi_{12}}{\psi_2^3} - \frac{\Phi_{22}\psi_1}{\psi_2^4}\right)\sum_{i=0}^{n-2k_n+1}\overline{Y}_i^2\sum_{j=i+k_n}^{i+2k_n-1} r_j^2 \\ &+ \frac{1}{n\theta^3}\left(\frac{\Phi_{11}}{\psi_2^2} - 2\frac{\Phi_{12}\psi_1}{\psi_2^3} + \frac{\Phi_{22}\psi_1^2}{\psi_2^4}\right)\sum_{i=0}^{n-2k_n+1} r_i^2 r_{i+2}^2. \end{aligned} \tag{5}$$

Together with the CLT result (4), we have that

$$T_n \equiv \frac{n^{1/4}\left(PRV_n - \int_0^1 \sigma_s^2 ds\right)}{\sqrt{\hat{V}_n}} \to^{st} N(0,1).$$

We can use this feasible asymptotic distribution result to build confidence intervals for integrated volatility. In particular, a two-sided feasible $100(1-\alpha)\%$ level interval for $\int_0^1 \sigma_s^2 ds$ is given by:

$$IC_{Feas,1-\alpha} = \left(PRV_n - z_{1-\alpha/2}n^{-1/4}\sqrt{\hat{V}_n}, PRV_n + z_{1-\alpha/2}n^{-1/4}\sqrt{\hat{V}_n}\right),$$

where $z_{1-\alpha/2}$ is such that $\Phi\left(z_{1-\alpha/2}\right) = 1-\alpha/2$, and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. For instance, $z_{0.975} = 1.96$ when $\alpha = 0.05$.

3 The bootstrap

The goal of this section is to propose a bootstrap method that can be used to consistently estimate the distribution of $n^{1/4}\left(PRV_n - \int_0^1 \sigma_s^2 ds\right)$. This justifies the construction of bootstrap percentile confidence intervals for integrated volatility. Although such intervals do not promise asymptotic refinements over confidence intervals based on the asymptotic mixed normal approximation (given by $IC_{Feas,1-\alpha}$), they avoid the need to explicitly estimate the asymptotic variance of the pre-averaged estimator. When the variance estimator is hard to compute (as it is the case here), it is not always clear that estimating the variance is beneficial in small samples. Thus, bootstrap percentile intervals are a very attractive method in these cases.

Gonçalves and Meddahi (2009) proposed bootstrap methods for realized volatility in the absence of market microstructure noise. In their ideal setting, intraday returns r_i are (conditionally on the volatility path) independent, but possibly heteroskedastic due to stochastic volatility, thus motivating the use of a wild bootstrap method.

When intraday returns are contaminated by market microstructure noise, they are no longer conditionally independent, as in Gonçalves and Meddahi (2009). This implies that the wild bootstrap is no longer valid when applied to r_i. Instead, a block bootstrap method applied to the intraday returns would seem appropriate.

One complication arises in this context: the statistic of interest is not symmetric in the observations and the block bootstrap generates blocks of observations that are conditionally independent. In particular, since the first term in PRV_n is an average of the squared pre-averaged returns $\bar{Y}_i^2$, it depends on all the products of intraday returns inside blocks of size k_n. If we generate block bootstrap intraday returns, these will be independent between blocks, implying that the bootstrap statistic may look at many pairs of intraday returns that are independent in the bootstrap world. This not only renders the analysis very complicated but can induce biases in the bootstrap estimator. To avoid this problem when dealing with statistics that are not symmetric in the underlying observations, Künsch (1989), Politis and Romano (1992) and Bühlmann and Künsch (1995) studied the "blocks of blocks" bootstrap, where one applies the block bootstrap to appropriately pre-specified blocks of observations. In our context, the blocks of blocks bootstrap consists of applying a traditional block bootstrap to the squared pre-averaged returns $\bar{Y}_i^2$. As we will see next, this approach is asymptotically valid only when volatility is constant. The reason is that when volatility is stochastic, squared pre-averaged returns are not only dependent but also heterogeneous. The block bootstrap does not capture this heterogeneity unless volatility is constant[1]. In order to capture both the time dependence and the heterogeneity in $\bar{Y}_i^2$, we propose a novel bootstrap procedure that combines the wild bootstrap with the block bootstrap.

Although the consistent estimator of integrated volatility is PRV_n, only the first term in PRV_n drives the variance of the limiting distribution of PRV_n. In particular, as Jacod et al. (2009) have shown, the second term is a bias correction term which does not contribute to the asymptotic variance (it only ensures that the estimator is well centered at the integrated volatility). For this reason, our proposal is to bootstrap only the first contribution to PRV_n,

$$\widetilde{PRV}_n = \frac{1}{\psi_2 k_n} \sum_{i=0}^{n-k_n+1} \bar{Y}_i^2.$$

This statistic depends only on the pre-averaged returns, to which we apply a particular bootstrap scheme. More specifically, let $\left\{\bar{Y}_i^* : i = 0, 1, \ldots, n - k_n + 1\right\}$ denote a bootstrap sample

[1]See Gonçalves and White (2002) for a discussion of the impact of mean heterogeneity on the validity of the block bootstrap for the sample mean.

from $\left\{\bar{Y}_i : i = 0, 1, \ldots, n - k_n + 1\right\}$. The bootstrap analogue of $\widetilde{PRV}_n$ is

$$\widetilde{PRV}_n^* = \frac{1}{\psi_2 k_n} \sum_{i=0}^{n-k_n+1} \bar{Y}_i^{*2}.$$

Since we do not incorporate a bias correction term in the bootstrap world, we center $\widetilde{PRV}_n^*$ around $E^*\left(\widetilde{PRV}_n^*\right)$. Thus, we use the bootstrap distribution of $n^{1/4}\left(\widetilde{PRV}_n^* - E^*\left(\widetilde{PRV}_n^*\right)\right)$ as an estimator of the distribution of $n^{1/4}\left(PRV_n - \int_0^1 \sigma_s^2 ds\right)$.

Next, we consider the blocks of blocks bootstrap approach applied to $\widetilde{PRV}_n$ and show that it is asymptotically invalid when volatility is time-varying. This motivates a new bootstrap method that combines the wild bootstrap with the block bootstrap, which we study in the last subsection.

3.1 The blocks of blocks bootstrap

To describe this approach, let $N_n = n - k_n + 2$ denote the total number of pre-averaged returns and let b_n denote the block size. We suppose that $N_n = J_n \cdot b_n$, so that J_n denotes the number of blocks of size b_n one needs to draw to get $N_n = n - k_n + 2$ bootstrap observations. The blocks of blocks bootstrap generates a bootstrap resample $\left\{\bar{Y}_{i-1}^* : i = 1, \ldots, N_n\right\}$ by applying the moving blocks bootstrap of Künsch (1989) to the scaled pre-averaged returns $\left\{\bar{Y}_{i-1} : i = 1, \ldots, N_n\right\}$.

Letting $I_1, \ldots, I_{J_n}$ be i.i.d. random variables distributed uniformly on $\{0, 1, \ldots, N_n - b_n\}$, we set

$$\bar{Y}_{i-1+(j-1)b_n}^* = \bar{Y}_{i-1+I_j} \text{ for } 1 \leq j \leq J_n \text{ and } 1 \leq i \leq b_n.$$

The bootstrap analogue of $\widetilde{PRV}_n$ is

$$\widetilde{PRV}_n^* = \frac{1}{\psi_2 k_n} \sum_{i=1}^{N_n} \bar{Y}_{i-1}^{*2} = \frac{1}{J_n} \sum_{j=1}^{J_n} \left(\frac{1}{b_n} \sum_{i=1}^{b_n} \underbrace{\frac{N_n}{k_n} \frac{1}{\psi_2} \bar{Y}_{I_j+i-1}^2}_{\equiv Z_{I_j+i}} \right),$$

where we let $Z_i \equiv \frac{N_n}{k_n} \frac{1}{\psi_2} \bar{Y}_{i-1}^2$. Note that in our setup, $\bar{Y}_i = \bar{X}_i + \bar{\epsilon}_i = O_P\left(n^{-1/4}\right)$ given that k_n is such that $k_n/\sqrt{n} = \theta + o\left(n^{-1/4}\right)$. This implies that $\bar{Y}_{i-1}^2 = O_P\left(n^{-1/2}\right)$ and therefore $Z_i = \frac{n-k_n+2}{k_n} \frac{1}{\psi_2} \bar{Y}_{i-1}^2$ is $O_P(1)$.

We can easily show that

$$E^*\left(\widetilde{PRV}_n^*\right) = \frac{1}{J_n} \sum_{j=1}^{J_n} E^*\left(\frac{1}{b_n} \sum_{i=1}^{b_n} Z_{I_j+i}\right) = \frac{1}{N_n - b_n + 1} \sum_{j=0}^{N_n-b_n} \left(\frac{1}{b_n} \sum_{i=1}^{b_n} Z_{j+i}\right).$$

Similarly,

$$
\begin{aligned}
V_n^* &\equiv Var^*\left(n^{1/4}\widetilde{PRV}_n^*\right) = \sqrt{n}E^*\left[\left(\frac{1}{J_n}\sum_{j=1}^{J_n}\frac{1}{b_n}\sum_{i=1}^{b_n}\left(Z_{I_j+i} - E^*\left(\widetilde{PRV}_n^*\right)\right)\right)^2\right] \\
&= \sqrt{n}\frac{1}{J_n}E^*\left(\frac{1}{b_n}\sum_{i=1}^{b_n}\left(Z_{I_1+i} - E^*\left(\widetilde{PRV}_n^*\right)\right)\right)^2 \\
&= \sqrt{n}\frac{b_n}{N_n}\frac{1}{N_n - b_n + 1}\sum_{j=0}^{N_n-b_n}\left(\frac{1}{b_n}\sum_{i=1}^{b_n}\left(Z_{j+i} - E^*\left(\widetilde{PRV}_n^*\right)\right)\right)^2. \qquad (6)
\end{aligned}
$$

Our next result studies the convergence of V_n^* when $b_n = (p+1)\,k_n$, for $p \geq 1$.

Lemma 3.1 *Suppose Assumption 1 holds and $k_n \to \infty$ as $n \to \infty$ such that Assumption 2 holds. Let $V_n^* \equiv Var^*\left(n^{1/4}\widetilde{PRV}_n^*\right)$ denote the moving blocks bootstrap variance of $n^{1/4}\widetilde{PRV}_n^*$ based on a block length equal to b_n. Then,*

a) *If $b_n = (p+1)\,k_n \to \infty$ and $p \geq 1$ is fixed,*

$$p\lim_{n\to\infty} V_n^* = V_p + B_p,$$

where

$$V_p = \int_0^1 \gamma^2(p)_t\,dt$$

with

$$\gamma^2(p)_t = \frac{4}{\psi_2^2}\left[\left(\Phi_{22} + \frac{1}{p+1}\Psi_{22}\right)\theta\sigma_t^4 + 2\left(\Phi_{12} + \frac{1}{p+1}\Psi_{12}\right)\frac{\sigma_t^2\alpha_t}{\theta} + \left(\Phi_{11} + \frac{1}{p+1}\Psi_{11}\right)\frac{\alpha_t^2}{\theta^3}\right],$$

and

$$B_p = \theta\,(p+1)\left[\int_0^1\left(\sigma_t^2 + \frac{\psi_1}{\theta^2\psi_2}\alpha_t\right)^2 dt - \left(\int_0^1\left(\sigma_t^2 + \frac{\psi_1}{\theta^2\psi_2}\alpha_t\right)dt\right)^2\right].$$

b) When σ is constant, $B_p = 0$ for any $p \geq 1$.

c) If $p \to \infty$ (i.e. $b_n/k_n = p+1 \to \infty$) such that $b_n/n \to 0$, then $V_p \to V \equiv \lim_{n\to\infty} Var\left(n^{1/4}PRV_n\right)$, so that $p\lim_{n\to\infty} V_n^* = V$ if σ is constant and $p\lim_{n\to\infty} V_n^* = \infty$ otherwise.

Part a) of Lemma 3.1 shows that when the bootstrap block size b_n is a fixed proportion of the pre-averaging block size k_n, the blocks of blocks bootstrap variance converges in probability to $V_p + B_p$, where B_p is a bias term due to the fact that volatility is time-varying. When σ is constant, B_p is equal to zero for any value of p. If $p \to \infty$ (i.e. if $b_n/k_n \to \infty$ as $n \to \infty$), then $V_p \to V$, the asymptotic variance of $n^{1/4}PRV_n$. Therefore, under this condition and assuming that σ is constant, we obtain the consistency of V_n^* towards V. If σ is stochastic and $p \to \infty$, then V_n^* diverges to infinity since $B_p \to \infty$ as $p \to \infty$.

Lemma 3.1 shows that the blocks of blocks bootstrap is consistent for the variance of PRV_n only under constant volatility and if we let the bootstrap block size b_n grow at a faster rate than the pre-averaging block size k_n. This result is related to a consistency result of the blocks of blocks bootstrap established in Bühlmann and Künsch (1995). As they showed, when the statistic of interest is an average of smooth functions of blocks of consecutive stationary strong mixing observations of size k_n, where k_n tends to infinity, the crucial condition for the block bootstrap to be valid is that the block size b_n grows at a faster rate than k_n. This is because the blocks over k_n observations (which in our case correspond to the pre-averaged returns) are strongly dependent for $|i-j| \leq k_n$, where $k_n \to \infty$, and b_n must be large enough to capture this dependence. Bühlmann and Künsch (1995) consider observations generated from a stationary strong mixing process and therefore they do not find any bias problem related to heterogeneity. Nevertheless, this becomes a problem in our context when volatility is stochastic. Therefore, a different bootstrap method is required to handle both the time dependence and the heterogeneity of pre-averaged returns.

3.2 The wild blocks of blocks bootstrap

In this section, we propose and study the consistency of a novel bootstrap method for pre-averaged returns based on overlapping blocks of k_n intraday returns. It combines the blocks of blocks bootstrap with the wild bootstrap and in this manner gets rid of the bias term B_p associated with the blocks of blocks bootstrap variance V_n^* in (6).

As in the previous section, for $p \geq 1$, let $b_n = (p+1)\,k_n$, and assume that J_n is such that $J_n \cdot b_n = N_n$. Let $\eta_1, \ldots, \eta_{J_n}$ be i.i.d. random variables whose distribution is independent of the original sample. Denote by $\mu_q^* = E^*\left(\eta_j^q\right)$ its q-th order moments. For $j = 1, \ldots, J_n$, let

$$\bar{B}_j = \frac{1}{b_n} \sum_{i=1}^{b_n} \bar{Y}_{i-1+(j-1)b_n}^2$$

denote the block average of the squared pre-averaged returns $\bar{Y}_{i-1+(j-1)b_n}^2$ for block j. We then generate the bootstrap pre-averaged squared returns as follows,

$$\bar{Y}_{i-1+(j-1)b_n}^{*2} = \bar{B}_{j+1} + \left(\bar{Y}_{i-1+(j-1)b_n}^2 - \bar{B}_{j+1}\right)\eta_j, \text{ for } 1 \leq j \leq J_n - 1 \text{ and for } 1 \leq i \leq b_n. \quad (7)$$

For the last block $j = J_n$, $\bar{B}_{j+1}$ is not available and therefore we let

$$\bar{Y}_{i-1+(j-1)b_n}^{*2} = \bar{B}_j + \left(\bar{Y}_{i-1+(j-1)b_n}^2 - \bar{B}_j\right)\eta_j, \text{ for } 1 \leq i \leq b_n. \quad (8)$$

Our method is related to the wild bootstrap approach of Wu (1986) and Liu (1988). More specifically, in Wu (1986) and Liu (1988), the statistic of interest is $\bar{X}_n$, where X_i is independently but heterogeneously distributed with mean μ_i and variance σ_i^2. Their wild bootstrap generates X_i^* as

$$X_i^* = \bar{X}_n + \left(X_i - \bar{X}_n\right)\eta_i, \text{ for } 1 \leq i \leq n,$$

where η_i is i.i.d. $(0,1)$. Liu (1988) shows that the bootstrap distribution of $\sqrt{n}\left(\bar{X}_n^* - \bar{X}_n\right)$ is consistent for the distribution of $\sqrt{n}\left(\bar{X}_n - \bar{\mu}_n\right)$, where $\bar{\mu}_n = n^{-1}\sum_{i=1}^n \mu_i$, provided $\frac{1}{n}\sum_{i=1}^n \left(\mu_i - \bar{\mu}_n\right)^2 \to 0$ (and some other regularity conditions).

Our bootstrap method can be seen as a generalization of the wild bootstrap of Wu (1986) and Liu (1988) to the k_n-dependent case. In particular, here the statistic of interest is an average of blocks of observations of size k_n,

$$\widetilde{PRV}_n = \frac{1}{N_n}\sum_{i=1}^{N_n} Z_i,$$

where $Z_i \equiv \frac{N_n}{k_n}\frac{1}{\psi_2}\bar{Y}_{i-1}^2$ has time-varying moments and is k_n-dependent (conditionally on X), i.e. Z_i is independent of Z_j for all $|i-j| > k_n$.

To preserve the serial dependence, we divide the data into J_n non-overlapping blocks of size b_n and generate the bootstrap observations within a given block j using the same external random variable η_j. This preserves the dependence within each block. When there is no dependence, we can take $b_n = 1$, in which case our bootstrap method amounts to Liu's wild bootstrap with one difference: instead of centering each bootstrap observation Z_i^* around the overall mean $\widetilde{PRV}_n$, we center Z_i^* around Z_{i+1}. The reason for the new centering is that μ_i in our context does not satisfy Liu's condition $\frac{1}{n}\sum_{i=1}^n \left(\mu_i - \bar{\mu}_n\right)^2 \to 0$ (unless volatility is constant). Hence centering around $\widetilde{PRV}_n$ does not work here. Instead, we show that centering around Z_{i+1} yields an asymptotically valid bootstrap method for $\widetilde{PRV}_n$ even when volatility is stochastic.

The bootstrap data generating process (7) and (8) yields a bootstrap sample $\left\{\bar{Y}_0^{*2}, \ldots, \bar{Y}_{N_n-1}^{*2}\right\}$ which we use to compute

$$PRV_n^* = \frac{1}{\psi_2 k_n}\sum_{i=1}^{N_n} \bar{Y}_{i-1}^{*2},$$

the wild blocks of blocks bootstrap analogue of $\widetilde{PRV}_n$. Let

$$\bar{B}_j^* = \frac{1}{b_n}\sum_{i=1}^{b_n} \bar{Y}_{i-1+(j-1)b_n}^{*2}$$

be the bootstrap analogue of $\bar{B}_j$. Given (7), we have that for $j = 1, \ldots, J_n - 1$,

$$\bar{B}_j^* = \bar{B}_{j+1} + \left(\bar{B}_j - \bar{B}_{j+1}\right)\eta_j,$$

whereas from (8), $\bar{B}_j^* = \bar{B}_j$ for $j = J_n$. This implies that we can write

$$\begin{aligned} PRV_n^* &= \frac{b_n}{\psi_2 k_n} \sum_{j=1}^{J_n} \frac{1}{b_n} \sum_{i=1}^{b_n} \bar{Y}_{i-1+(j-1)b_n}^{*2} = \frac{b_n}{\psi_2 k_n} \sum_{j=1}^{J_n-1} \bar{B}_j^* + \frac{b_n}{\psi_2 k_n} \bar{B}_{J_n}^* \\ &= \frac{b_n}{\psi_2 k_n} \sum_{j=1}^{J_n-1} \left[\bar{B}_{j+1} + \left(\bar{B}_j - \bar{B}_{j+1} \right) \eta_j \right] + \frac{b_n}{\psi_2 k_n} \bar{B}_{J_n}. \end{aligned}$$

We can now easily obtain the bootstrap mean and variance of PRV_n^*. In particular,

$$E^* \left(PRV_n^* \right) = \frac{b_n}{\psi_2 k_n} \left(\sum_{j=1}^{J_n-1} \bar{B}_{j+1} + \bar{B}_{J_n} \right) + \frac{b_n}{\psi_2 k_n} \sum_{j=1}^{J_n-1} \left(\bar{B}_j - \bar{B}_{j+1} \right) E^* \left(\eta_j \right),$$

and

$$V_n^* \equiv Var^* \left(n^{1/4} PRV_n^* \right) = \frac{n^{1/2} b_n^2}{\psi_2^2 k_n^2} \sum_{j=1}^{J_n-1} \left(\bar{B}_j - \bar{B}_{j+1} \right)^2 Var^* \left(\eta_j \right).$$

Our next result studies the convergence of V_n^* when $b_n = (p+1) k_n$ and p is either fixed such that $p \geq 1$, or $p \to \infty$.

Lemma 3.2 *Suppose Assumption 1 holds and $k_n \to \infty$ as $n \to \infty$ such that Assumption 2 holds. Let $V_n^* \equiv Var^* \left(n^{1/4} PRV_n^* \right)$ denote the wild blocks of blocks bootstrap variance of $n^{1/4} PRV_n^*$ based on a block length equal to b_n and external random variables $\eta_j \sim$ i.i.d. with mean $E^* \left(\eta_j \right)$ and variance $Var^* \left(\eta_j \right)$. Then,*

a) *If $b_n = (p+1) k_n \to \infty$ and p is fixed,*

$$p \lim_{n \to \infty} V_n^* = 2 Var^* \left(\eta_j \right) V_p + O_P \left(\frac{1}{p} \right),$$

where V_p is as defined in Lemma 3.1.

b) If $p \to \infty$ (i.e. $b_n / k_n = p + 1 \to \infty$) such that $b_n / n \to 0$ and $Var^* \left(\eta_j \right) = 1/2$, then $V_p \to V \equiv \lim_{n \to \infty} Var \left(n^{1/4} PRV_n \right)$ so that $p \lim_{n \to \infty} V_n^* = V$.

This result shows that if we let b_n grow faster than k_n and $Var^* \left(\eta_j \right) = 1/2$, the wild blocks bootstrap variance estimator is consistent for the asymptotic variance of PRV_n under Assumptions 1 and 2. Given the consistency of the bootstrap variance estimator, we can now prove the consistency of the bootstrap distribution of $n^{1/4} \left(PRV_n^* - E^* \left(PRV_n^* \right) \right)$.

Theorem 3.1 *Suppose Assumption 1 holds and $k_n \to \infty$ as $n \to \infty$ such that Assumption 2 holds. Let PRV_n^* be the pre-averaged realized volatility estimator based on a block length equal to b_n and an external random variable $\eta_j \sim$ i.i.d. $\left(E^* \left(\eta_j \right), Var^* \left(\eta_j \right) \right)$ such that $Var^* \left(\eta_j \right) = \frac{1}{2}$,* *and for any* $\delta > 0$ $E^* \left|\eta_j\right|^{2+\delta} \leq \Delta < \infty$*. If* b_n *is such that* $b_n = (p+1)\, k_n$*,* $b_n/n \to 0$ *and* $p \to \infty$*, then*

$$\sup_{x\in\mathbb{R}} \left| P^* \left(n^{1/4} \left(PRV_n^* - E^* \left(PRV_n^*\right)\right) \leq x\right) - P\left(n^{1/4}\left(PRV_n - \int_0^1 \sigma_s^2 ds\right) \leq x\right)\right| \to^P 0 \text{ as } n \to \infty.$$

4 Monte Carlo results

In this section, we compare the finite sample performance of the bootstrap with the feasible asymptotic theory for confidence intervals of integrated volatility.

We consider two data generating processes in our simulations. First, following Zhang et al. (2005), we use the one-factor stochastic volatility (SV1F) model of Heston (1993) as our data-generating process, i.e.

$$dX_t = (\mu - \nu_t/2)\, dt + \sigma_t dB_t,$$

and

$$d\nu_t = \kappa\left(\alpha - \nu_t\right) dt + \gamma\left(\nu_t\right)^{1/2} dW_t,$$

where $\nu_t = \sigma_t^2$, and we assume $Corr(B, W) = \rho$. The parameter values are all annualized. In particular, we let $\mu = 0.05/252$, $\kappa = 5/252$, $\alpha = 0.04/252$, $\gamma = 0.05/252$, $\rho = -0.5$. The size of the market microstructure noise is an important parameter. We follow Barndorff-Nielsen et al. (2008) and model the noise magnitude as $\xi^2 = \omega^2/\sqrt{\int_0^1 \sigma_s^4 ds}$. We fix ξ^2 equal to 0.0001, 0.001 and 0.01 and let $\omega^2 = \xi^2\sqrt{\int_0^1 \sigma_s^4 ds}$. These values are motivated by the empirical study of Hansen and Lunde (2006), who investigate 30 stocks of the Dow Jones Industrial Average.

We also consider the two-factor stochastic volatility (SV2F) model analyzed by Barndorff-Nielsen et al. (2008), where [2]

$$\begin{aligned}
dX_t &= \mu dt + \sigma_t dB_t, \\
\sigma_t &= s\text{-}exp\left(\beta_0 + \beta_1 \tau_{1t} + \beta_2 \tau_{2t}\right), \\
d\tau_{1t} &= \alpha_1 \tau_{1t} dt + dB_{1t}, \\
d\tau_{2t} &= \alpha_2 \tau_{2t} dt + \left(1 + \phi \tau_{2t}\right) dB_{2t}, \\
corr\left(dW_t, dB_{1t}\right) &= \varphi_1,\ corr\left(dW_t, dB_{2t}\right) = \varphi_2.
\end{aligned}$$

We follow Huang and Tauchen (2005) and set $\mu = 0.03$, $\beta_0 = -1.2$, $\beta_1 = 0.04$, $\beta_2 = 1.5$, $\alpha_1 = -0.00137$, $\alpha_2 = -1.386$, $\phi = 0.25$, $\varphi_1 = \varphi_2 = -0.3$. We initialize the two factors at the start of each interval by drawing the persistent factor from its unconditional distribution, $\tau_{10} \sim N\left(0, \frac{-1}{2\alpha_1}\right)$, and by starting the stronlgly mean-reverting factor at zero.

[2] The function s-exp is the usual exponential function with a linear growth function splined in at high values of its argument: $s\text{-}exp(x) = exp(x)$ if $x \leq x_0$ and $s\text{-}exp(x) = \frac{exp(x_0)}{\sqrt{x_0 - x_0^2 + x^2}}$ if $x > x_o$, with $x_0 = log(1.5)$.

We simulate data for the unit interval $[0,1]$ and normalize one second to be $1/23400$, so that $[0,1]$ is thought to span 6.5 hours. The observed Y process is generated using an Euler scheme. We then construct the $\frac{1}{n}$-horizon returns $r_i \equiv Y_{i/n} - Y_{(i-1)/n}$ based on samples of size n.

We use two different values of θ: $\theta = 1/3$, as in Jacod et al. (2009), and $\theta = 1$, as in Christensen, Kinnebrock and Podolskij (2010). The latter value corresponds to a conservative choice of k_n. We also follow the literature and use the weight function $g(x) = \min(x, 1-x)$ to compute the pre-averaged returns.

In order to reduce finite sample biases associated with Riemann integrals, we follow Jacod et al. (2009) and Hautsch and Podolskij (2013) and use the finite sample adjustments version of the pre-averaged realized volatility estimator,

$$PRV_n^a = \left(1 - \frac{\psi_1^{k_n}}{2n\theta^2\psi_2^{k_n}}\right)^{-1} \left(\frac{n}{n-k_n+2}\frac{1}{\psi_2^{k_n}k_n}\sum_{i=0}^{n-k_n+1}\bar{Y}_i^2 - \frac{\psi_1^{k_n}}{2n\theta^2\psi_2^{k_n}}\sum_{i=1}^{n} r_i^2\right),$$

where $\psi_1^{k_n} = k_n \sum_{i=1}^{k_n}\left(g\left(\frac{i}{k_n}\right) - g\left(\frac{i-1}{k_n}\right)\right)^2$ and $\psi_2^{k_n} = \frac{1}{k_n}\sum_{i=1}^{k_n} g^2\left(\frac{i}{k_n}\right)$. Similarly, $\hat{V}_n$ as defined in (5) replaces Φ_{11}, Φ_{12} and Φ_{22} by their Riemann approximations,

$$\Phi_{11}^{k_n} = k_n\left(\sum_{i=1}^{k_n}\left(\phi_1^{k_n}(j)\right)^2 - \frac{1}{2}\left(\phi_1^{k_n}(0)\right)^2\right), \quad \Phi_{12}^{k_n} = \frac{1}{k_n}\left(\sum_{i=1}^{k_n}\phi_1^{k_n}(j)\,\phi_2^{k_n}(j) - \frac{1}{2}\phi_1^{k_n}(0)\,\phi_2^{k_n}(0)\right), \quad \text{and}$$

$$\Phi_{22}^{k_n} = \frac{1}{k_n^3}\left(\sum_{i=1}^{k_n}\left(\phi_2^{k_n}(j)\right)^2 - \frac{1}{2}\left(\phi_2^{k_n}(0)\right)^2\right),$$

where

$$\phi_1^{k_n}(j) = k_n\sum_{i=j+1}^{k_n-1}\left(g\left(\frac{i-1}{k_n}\right) - g\left(\frac{i}{k_n}\right)\right)\left(g\left(\frac{i-j-1}{k_n}\right) - g\left(\frac{i-j}{k_n}\right)\right), \quad \text{and}$$

$$\phi_2^{k_n}(j) = \sum_{i=j+1}^{k_n-1} g\left(\frac{i}{k_n}\right) - g\left(\frac{i-j}{k_n}\right).$$

Tables 1 and 2 give the actual rates of 95% confidence intervals of integrated volatility for the SV1F and the SV2F models, respectively, computed over 10,000 replications. Results are presented for eight different samples sizes: n = 23400, 11700, 7800, 4680, 1560, 780, 390 and 195, corresponding to "1-second", "2-second", "3-second", "5-second", "15-second", "30-second", "1-minute" and "2-minute" frequencies.

In our simulations, bootstrap intervals use 999 bootstrap replications for each of the 10,000 Monte Carlo replications. We consider the bootstrap percentile method computed at the 95% level. To generate the bootstrap data we use the following external random variables $\eta_j \sim$ i.i.d. $N(0, 1/2)$. The choice of the bootstrap block size is critical. We follow Politis, Romano and Wolf (1999) and use the Minimum Volatility Method to choose the bootstrap block. Details of the algorithm are given in Appendix A.

For the two models, all intervals tend to undercover. The degree of undercoverage is especially large for smaller values of n, when sampling is not too frequent. The SV2F model exhibits overall larger coverage distortions than the SV1F model, for all sample sizes. Results are sensitive to the value of the tuning parameter θ. When $\theta = 1/3$, larger market microstructure effects induce larger coverage distortions. In particular, the coverage distortions are very important when $\xi^2 = 0.01$ in comparison to the case where market microstructure effects are moderate or negligible ($\xi^2 = 0.001$ and $\xi^2 = 0.0001$). This reflects the fact that for this value of θ, k_n is not sufficiently large to allow pre-averaging to remove the market microstructure bias. The pre-averaged estimator is biased in finite samples and this explains the finite sample distortions. In contrast, for the conservative choice of k_n, results are not very sensitive to the noise magnitude. The reason is that the larger is the block size over which the pre-averaging is done, the smaller is the impact of the noise.

In all cases, the bootstrap outperforms the existing first order asymptotic theory. As expected, the average chosen block size is larger for larger sample sizes, but our results show that it is not sensitive to the noise magnitude. This is because the noise magnitude is almost irrelevant for the intensity of the autocorrelation of the square pre-averaged returns (as confirmed by simulations not reported here).

5 Empirical results

In this section, we implement the wild blocks of blocks bootstrap on high frequency data and compare it to the existing feasible asymptotic procedure of Jacod et al. (2009). The data consists of transaction log prices of General Electric (GE) shares carried out on the New York Stock Exchange (NYSE) in October 2011. Our procedure for cleaning the data is exactly identical to that used by Barndorff-Nielsen et al. (2008) (for further details see Barndorff-Nielsen et al. (2009)). For each day, we consider data from the regular exchange opening hours from time stamped between 9:30 a.m. until 4 p.m.

We implement the pre-averaged realized volatility estimator of Jacod et al. (2009) on returns recorded every S transactions, where S is selected each day so that there are approximately 1493 observations a day. This means that on average these returns are recorded roughly every 15 seconds. Table 3 in Appendix A provides the number of transactions per day and the sample size for the pre-averaged returns.

To implement the pre-averaged realized volatility estimator, we select the tunning parameter θ by following the conservative rule ($\theta = 1$, implying that $k_n = \sqrt{n}$). To choose the block size b_n, we follow Politis, Romano and Wolf (1999) and use the Minimum Volatility Method (see Appendix A for details).

Figure 1 in Appendix A shows daily 95% confidence intervals (CIs) for integrated volatility using both methods, the wild blocks of blocks bootstrap and the existing feasible asymptotic procedure of Jacod et al. (2009). The confidence intervals based on the bootstrap method are usually wider than the confidence intervals using the feasible asymptotic theory.[3] This is especially true in periods with large volatility. To gain further insight on the behavior of our intervals for these periods, we implemented the test for jumps of Barndorff- Nielsen and Shephard (2006) using a moderate sample size (2-minute sampling intervals). It turns out that these days often correspond to days on which there is evidence for jumps (in particular for the 13, 17, 20 and 26 of October 2011). Since neither of the two types of intervals are valid in the presence of jumps, further analysis should be pursued for these particular days. In particular, we should rely on estimation methods that are robust to jumps such as the pre-averaged multipower variation method proposed by Podolskij and Vetter (2009) or the quantile estimation method of Christensen, Oomen, and Podolskij (2010).

6 Conclusion

In this paper, we propose the bootstrap as a method of inference for integrated volatility in the context of the pre-averaged realized volatility estimator proposed by Jacod et al. (2009). We show that the "blocks of blocks" bootstrap method suggested by Politis and Romano (1992) is valid in this context only when volatility is constant. This is due to the heterogeneity of the squared pre-averaged returns when volatility is stochastic.

To simultaneously handle the dependence and heterogeneity of the pre-averaged returns, we propose a novel bootstrap procedure that combines the wild and the blocks of blocks bootstrap. We provide a set of conditions under which this method is asymptotically valid to first order. Our Monte Carlo simulations show that the wild blocks of blocks bootstrap improves the finite sample properties of the existing first order asymptotic theory. The empirical results suggest that this bootstrap method is generally more accurate than the existing feasible approach of Jacod et al. (2009). In future work, we plan to generalize the wild blocks of blocks bootstrap for inference on multivariate integrated volatility as considered by Christensen, Kinnebrock and Podolskij (2010). Bootstrap variance-covariances matrices are naturally positive semi-definite, which is very important for empirical applications.

[3]Nevertheless, as our Monte Carlo simulations showed, the latter typically have undercoverage problems whereas the bootstrap intervals have coverage rates closer to the desired level. Therefore if the goal is to control the coverage probability, shorter intervals are not necessarily better. The figures also show a lot of variability in the daily estimate of integrated volatility.

Appendix A: Simulation and empirical results

Here we describe the Minimum Volatility Method algorithm of Politis, Romano and Wolf (1999, Chapter 9) for choosing the block size b_n for a two-sided confidence interval.

Algorithm: Choice of the bootstrap block size by minimizing confidence interval volatility

(i) For $b = b_{small}$ to $b = b_{big}$ compute a bootstrap interval for IV at the desired confidence level, this resulting in endpoints $IC_{b,low}$ and $IC_{b,up}$.

(ii) For each b compute the volatility index VI_b as the standard deviation of the interval endpoints in a neighborhood of b. More specifically, for a smaller integer d, let VI_b equal to the standard deviation of the endpoints $\{IC_{b-d,low}, \ldots, IC_{b+d,low}\}$ plus the standard deviation of the endpoints $\{IC_{b-d,up}, \ldots, IC_{b+d,up}\}$, i.e.

$$VI_b \equiv \sqrt{\frac{1}{2d+1}\sum_{i=-d}^{d}\left(IC_{b+i,low} - \bar{IC}_{low}\right)^2} + \sqrt{\frac{1}{2d+1}\sum_{i=-d}^{d}\left(IC_{b+i,up} - \bar{IC}_{up}\right)^2},$$

where $\bar{IC}_{low} = \frac{1}{2d+1}\sum_{i=-d}^{d} IC_{b+i,low}$ and $\bar{IC}_{up} = \frac{1}{2d+1}\sum_{i=-d}^{d} IC_{b+i,up}$.

(iii) Pick the value b^* corresponding to the smallest volatility index and report $\{IC_{b^*,low}, IC_{b^*,up}\}$ as the final confidence interval.

To make the algorithm more computationally efficient, we have skipped a number of b values in regular fashion between b_{small} and b_{big}. We have considered only the values of b such that $b = pk_n$ where p is a fixed integer. We employ $b_{small} = 2k_n$, $b_{big} = \min(\theta\frac{N_n}{4}, 12k_n)$ and $d = 2$.

Tables 1 and 2 report the actual coverage rates for the feasible asymptotic theory approach and for our bootstrap methods using the optimal block size by minimizing confidence interval volatility. In Table 3 we provide some statistics of GE shares in January 2011.

Table 1: Coverage rates of nominal 95% intervals using $\theta = 1/3$

		SV1F			SV2F		
	n	CLT	Boot	Avg. block size	CLT	Boot	Avg. block size
$\xi^2 = 0.0001$							
	195	90.89	91.02	11.75	88.60	90.09	11.73
	390	91.52	91.74	21.09	90.32	90.98	22.16
	780	92.88	93.41	32.63	91.40	92.94	34.31
	1560	93.86	94.01	65.50	92.62	93.71	68.58
	4680	94.32	94.43	144.69	93.94	94.43	143.98
	7800	94.68	94.72	172.40	94.19	95.02	179.48
	11700	94.60	94.87	220.48	94.17	95.14	224.81
	23400	94.80	94.93	319.21	94.68	95.10	319.67
$\xi^2 = 0.001$							
	195	90.77	90.88	11.68	88.20	90.07	11.80
	390	91.14	91.43	20.71	89.31	90.21	21.78
	780	92.26	93.50	32.33	90.80	92.54	34.24
	1560	93.40	94.12	65.11	92.61	94.85	69.73
	4680	94.46	95.07	140.71	93.65	95.20	151.50
	7800	94.14	95.24	174.08	94.05	95.35	172.34
	11700	94.23	95.13	219.74	93.98	95.45	222.15
	23400	94.47	95.04	323.09	94.50	95.23	312.43
$\xi^2 = 0.01$							
	195	83.11	88.51	11.73	80.96	87.79	11.56
	390	84.45	91.16	20.68	83.91	89.98	21.81
	780	86.48	91.92	31.67	85.89	91.96	32.09
	1560	87.97	93.10	64.84	88.02	93.61	62.08
	4680	91.13	94.17	144.19	90.76	94.12	142.92
	7800	91.92	94.91	170.45	91.45	94.26	170.06
	11700	92.20	94.52	216.41	92.19	94.61	215.82
	23400	92.87	94.85	323.29	92.88	95.12	315.95

Notes: CLT-intervals based on the Normal; Boot-intervals based on the bootstrap. 10,000 Monte Carlo trials with 999 bootstrap replications each.

Table 2: Coverage rates of nominal 95% intervals using $\theta = 1$

		SV1F			SV2F		
	n	CLT	Boot	Avg. block size	CLT	Boot	Avg. block size
$\xi^2 = 0.0001$							
	195	89.48	90.10	35.90	84.50	87.12	36.27
	390	91.41	94.30	65.86	86.63	91.47	65.86
	780	92.81	94.98	132.22	88.71	92.10	124.99
	1560	93.57	95.13	262.24	90.39	93.92	235.76
	4680	94.19	95.45	517.12	91.50	94.20	451.02
	7800	94.27	95.12	682.52	92.76	95.00	594.68
	11700	94.06	95.50	804.62	93.15	94.81	713.17
	23400	94.39	95.48	1210.69	93.80	94.90	1063.81
$\xi^2 = 0.001$							
	195	89.05	92.19	35.90	84.41	87.60	35.92
	390	91.31	94.63	65.78	86.90	91.86	66.06
	780	92.96	94.76	132.78	88.57	92.80	124.15
	1560	93.66	95.37	265.00	90.34	94.30	237.96
	4680	94.12	95.52	514.43	92.03	94.51	458.33
	7800	94.21	95.16	688.04	92.32	94.88	582.40
	11700	94.17	95.18	806.15	92.98	95.01	719.93
	23400	94.35	95.11	1210.23	93.80	94.86	1062.43
$\xi^2 = 0.01$							
	195	88.42	92.18	35.81	84.07	88.62	35.97
	390	90.51	94.60	66.44	86.58	91.31	66.16
	780	92.17	95.12	132.58	88.52	92.87	125.22
	1560	93.35	95.15	264.96	90.01	94.40	243.92
	4680	93.77	95.60	515.74	91.72	95.23	471.10
	7800	94.28	95.72	671.84	92.76	95.20	593.08
	11700	94.16	95.24	808.00	93.03	95.40	732.35
	23400	94.26	95.18	1197.28	93.70	95.31	1081.40

Notes: CLT-intervals based on the Normal; Boot-intervals based on the bootstrap. 10,000 Monte Carlo trials with 999 bootstrap replications each.

Table 3: Summary statistics

Days	Trans	S	n
3 Oct	12613	9	1402
4 Oct	13782	9	1532
5 Oct	10628	7	1519
6 Oct	9991	7	1428
7 Oct	9785	7	1398
10 Oct	10660	7	1523
11 Oct	8588	6	1432
12 Oct	11160	7	1595
13 Oct	8649	6	1442
14 Oct	9261	6	1544
17 Oct	8530	6	1422
18 Oct	8751	6	1459
19 Oct	9023	6	1504
20 Oct	9251	6	1542
21 Oct	12513	8	1565
24 Oct	11642	8	1456
25 Oct	10919	8	1365
26 Oct	9249	6	1542
27 Oct	14598	9	1622
28 Oct	9405	6	1568
31 Oct	8871	6	1500

"Trans" denotes the number of transactions, n is the sample size used to calculate the pre-averaged realized volatility, we have sampled every Sth transaction price, so the period over which returns are calculated is roughly 15 seconds.

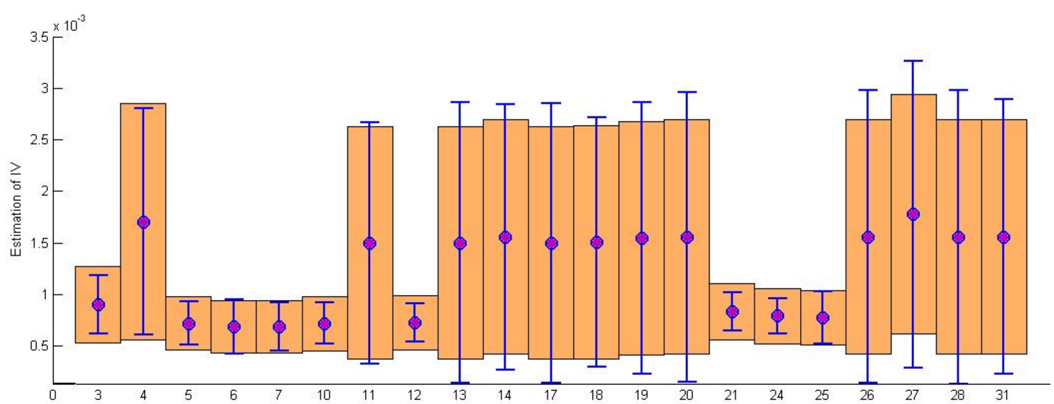

Figure 1: 95% Confidence Intervals (CI's) for the daily IV, for each regular exchange opening days in October 2011, calculated using the asymptotic theory of Jacod et al. (2009) (CI's with bars), and the wild blocks of blocks bootstrap method (CI's with lines). The pre-averaging realized volatility estimator is the middle of all CI's by construction. Days on the x-axis.

Appendix B: Proofs

As in Jacod et al. (2009), we assume throughout this Appendix that the processes a, σ and X are bounded processes satisfying (1) with a and σ adapted càdlàg processes. As Jacod et al. (2009) explain, this assumption simplifies the mathematical derivations without loss of generality (by a standard localization procedure detailed in Jacod (2008)). Formally, we derive our results under the following assumption.

Assumption 3. X satisfies equation (1) with a and σ adapted càdlàg processes such that a, σ, and X are bounded processes (implying that α is also bounded).

Notation

In the following, K denotes a constant which changes from line to line. Moreover, we follow Jacod et al. (2009) and use the following additional notation. We let

$$\bar{X}_i = \sum_{j=1}^{k_n} g\left(\frac{j}{k_n}\right)\left(X_{\frac{i+j}{n}} - X_{\frac{i+j-1}{n}}\right), \quad \bar{\epsilon}_i = \sum_{j=1}^{k_n} g\left(\frac{j}{k_n}\right)\left(\epsilon_{\frac{i+j}{n}} - \epsilon_{\frac{i+j-1}{n}}\right),$$

and note that $\bar{Y}_i = \bar{X}_i + \bar{\epsilon}_i$. In addition, we let

$$\begin{aligned} c_i &= \sum_{j=1}^{k_n} g\left(\frac{j}{k_n}\right)^2 \int_{\frac{i+j-1}{n}}^{\frac{i+j}{n}} \sigma_t^2 dt; \\ A_i &= E\left(\bar{\epsilon}_i^2 | X\right) = \sum_{j=0}^{k_n-1}\left(g\left(\frac{j+1}{k_n}\right) - g\left(\frac{j}{k_n}\right)\right)^2 \alpha_{(i+j)/n}; \text{ and} \\ \widetilde{Y}_i &= \bar{Y}_i^2 - A_i - c_i. \end{aligned}$$

Following Jacod et al. (2009), we also introduce the following random variables. For $j = 1, \ldots, J_n$, we let

$$\eta\left(p\right)_j = \frac{1}{\theta\psi_2\sqrt{n}}\zeta(p)_{(j-1)(p+1)k_n}, \text{ with } \zeta\left(p\right)_j = \sum_{i=j}^{j+(p+1)k_n-1} \widetilde{Y}_i,$$

where $p \geq 1$ is a fixed integer; $\eta\left(p\right)_j$ is the normalized sum of squared pre-averaged returns $\tilde{Y}_i$ over a block of size $b_n = (p+1)\, k_n$. Note that $\eta\left(p\right)_j$ is measurable with respect to $\mathcal{F}^n_{j(p+1)k_n}$, the sigma algebra generated by all $\mathcal{F}^0_{j(p+1)k_n/n}$-measurable random variables plus all variables Y_s, with $s < j\left(p+1\right)k_n$. Finally, we let

$$\beta(p)_i = \sup_{s,t \in \left[\frac{i}{n}, \frac{i+(p+1)k_n}{n}\right]} \left(|a_s - a_t| + |\sigma_s - \sigma_t| + |\alpha_s - \alpha_t|\right), \tag{9}$$

and

$$\gamma^2(p)_t = \frac{4}{\psi_2^2}\left(\left(\Phi_{22} + \frac{1}{p+1}\Psi_{22}\right)\theta\sigma_t^4 + 2\left(\Phi_{12} + \frac{1}{p+1}\Psi_{12}\right)\frac{\sigma_t^2\alpha_t}{\theta} + \left(\Phi_{11} + \frac{1}{p+1}\Psi_{11}\right)\frac{\alpha_t^2}{\theta^3}\right). \tag{10}$$

Our bootstrap estimators depend crucially on

$$\bar{B}_j \equiv \frac{1}{b_n}\sum_{i=1}^{b_n} \bar{Y}_{i-1+(j-1)b_n}^2 = \frac{1}{b_n}\sum_{i=(j-1)b_n}^{jb_n-1} \bar{Y}_i^2, \text{ for } j=1,\ldots,J_n,$$

where $J_n = N_n/b_n$ is the number of non-overlapping blocks of size b_n out of $N_n = n - k_n + 2$ observations on pre-averaged returns.

Our first result is instrumental in proving our bootstrap results.

Lemma B.1 *Suppose Assumptions 2 and 3 hold. Then, for all integer $p \geq 1$, and each $q > 0$, we have that*

a1) $\frac{1}{\sqrt{n}} E\left(\sum_{j=1}^{J_n} \beta\left(p\right)_{(j-1)(p+1)k_n}^q\right) \to 0.$

a2) $\frac{1}{\sqrt{n}} \sum_{j=1}^{J_n} \beta\left(p\right)_{(j-1)(p+1)k_n}^q \to^P 0.$

a3) $\frac{1}{\sqrt{n}} E\left(\sum_{j=1}^{J_n} E\left(\beta\left(p\right)_{(j-1)(p+1)k_n}^q |\mathcal{F}_{(j-1)(p+1)k_n}^n\right)\right) \to 0.$

a4) $\frac{1}{\sqrt{n}} \sum_{j=1}^{J_n} E\left(\beta\left(p\right)_{(j-1)(p+1)k_n}^q |\mathcal{F}_{(j-1)(p+1)k_n}^n\right) \to^P 0.$

a5) $\frac{1}{\sqrt{n}} \sum_{j=1}^{J_n} E\left(\beta\left(2p+1\right)_{(j-1)(p+1)k_n}^q |\mathcal{F}_{(j-1)(p+1)k_n}^n\right) \to^P 0.$

a6) $\frac{1}{\sqrt{n}} \sum_{j=1}^{J_n} \sqrt{E\left(\beta\left(p\right)_{(j-1)(p+1)k_n}^2 |\mathcal{F}_{(j-1)(p+1)k_n}^n\right)} \to^P 0.$

a7) $\frac{1}{\sqrt{n}} \sum_{j=1}^{J_n} \sqrt{E\left(\beta\left(2p+1\right)_{(j-1)(p+1)k_n}^2 |\mathcal{F}_{(j-1)(p+1)k_n}^n\right)} \to^P 0.$

Proof of Lemma B.1. Part a1). Given the definition of $\beta\left(p\right)_{(j-1)(p+1)k_n}$ we can write

$$\begin{aligned}
\beta\left(p\right)_{(j-1)(p+1)k_n} &\leq \sup_{s,t\in\left[\frac{(j-1)(p+1)k_n}{n},\frac{(j-1)(p+1)k_n+(p+1)k_n}{n}\right]}\left(|a_s - a_t|\right) \\
&\quad + \sup_{s,t\in\left[\frac{(j-1)(p+1)k_n}{n},\frac{(j-1)(p+1)k_n+(p+1)k_n}{n}\right]}\left(|\sigma_s - \sigma_t|\right) \\
&\quad + \sup_{s,t\in\left[\frac{(j-1)(p+1)k_n}{n},\frac{(j-1)(p+1)k_n+(p+1)k_n}{n}\right]}\left(|\alpha_s - \alpha_t|\right) \\
&\equiv \Gamma\left(a,p\right)_{(j-1)(p+1)k_n} + \Gamma\left(\sigma,p\right)_{(j-1)(p+1)k_n} + \Gamma\left(\alpha,p\right)_{(j-1)(p+1)k_n}.
\end{aligned}$$

Given that $\Gamma\left(a,p\right)_{(j-1)(p+1)k_n}, \Gamma\left(\sigma,p\right)_{(j-1)(p+1)k_n}$ and $\Gamma\left(\alpha,p\right)_{(j-1)(p+1)k_n}$ are strictly positive, for any $q > 0$, using the c-r inequality, we can write

$$\beta\left(p\right)_{(j-1)(p+1)k_n}^q \leq K\left(\Gamma\left(\sigma,p\right)_{(j-1)(p+1)k_n}^q + \Gamma\left(a,p\right)_{(j-1)(p+1)k_n}^q + \Gamma\left(\alpha,p\right)_{(j-1)(p+1)k_n}^q\right).$$

It follows that

$$
\begin{aligned}
n^{-1/2}E\left(\sum_{j=1}^{J_n}\beta\left(p\right)^{q}_{(j-1)(p+1)k_n}\right) &\leq Kn^{-1/2}E\left(\sum_{j=1}^{J_n}\Gamma\left(\sigma,p\right)^{q}_{(j-1)(p+1)k_n}\right)\\
&+Kn^{-1/2}E\left(\sum_{j=1}^{J_n}\Gamma\left(a,p\right)^{q}_{(j-1)(p+1)k_n}\right)\\
&+Kn^{-1/2}E\left(\sum_{j=1}^{J_n}\Gamma\left(\alpha,p\right)^{q}_{(j-1)(p+1)k_n}\right)=o\left(1\right),
\end{aligned}
$$

where we use Lemma 5.3 of Jacod, Podolskij and Vetter (2010) to show that each of the terms above are $o\left(1\right)$ (given that a, σ and α are càdlàg bounded processes).

Proof of Lemma B.1. Part a2). Note that given the result of part a1) of Lemma B.1, it is sufficient to show that $\frac{1}{n}E\left(\sum_{j=1}^{J_n}\beta\left(p\right)^{q}_{(j-1)(p+1)k_n}\right)^{2}\to 0$. By the c-r inequality,

$$
\frac{1}{n}E\left(\sum_{j=1}^{J_n}\beta\left(p\right)^{q}_{(j-1)(p+1)k_n}\right)^{2}\leq\frac{J_n}{n}E\left(\sum_{j=1}^{J_n}\beta\left(p\right)^{2q}_{(j-1)(p+1)k_n}\right)\leq K\frac{1}{\sqrt{n}}E\left(\sum_{j=1}^{J_n}\beta\left(p\right)^{2q}_{(j-1)(p+1)k_n}\right),
$$

which is $o\left(1\right)$ by part a1) of Lemma B.1 and given that $J_n=O\left(\sqrt{n}\right)$.

Proof of Lemma B.1. Part a3). Given the law of iterated expectations, the result follows directly from part a1) of Lemma **B.1.**

Proof of Lemma B.1. Part a4). The proof follows similarly as in part a2) of Lemma **B.1**, where we now consider the variable $E\left(\beta\left(p\right)^{q}_{(j-1)(p+1)k_n}|\mathcal{F}^{n}_{(j-1)(p+1)k_n}\right)$ in place of $\beta\left(p\right)^{q}_{(j-1)(p+1)k_n}$.

Proof of Lemma B.1. Part a5). Given the definition of $\beta(p)_i$, for any $p\geq 1$, such that $b_n=\left(p+1\right)k_n$ we can write

$$
\begin{aligned}
\frac{1}{\sqrt{n}}\sum_{j=1}^{J_n}E\left(\beta\left(2p+1\right)^{q}_{(j-1)b_n}|\mathcal{F}^{n}_{(j-1)b_n}\right) &= \frac{1}{\sqrt{n}}\sum_{j=1}^{\left[\frac{J_n}{2}\right]}E\left(\beta\left(2p+1\right)^{q}_{2(j-1)b_n}|\mathcal{F}^{n}_{2(j-1)b_n}\right)\\
&+\frac{1}{\sqrt{n}}\sum_{j=1}^{\left[\frac{J_n}{2}\right]}E\left(\beta\left(2p+1\right)^{q}_{(2(j-1)+1)b_n}|\mathcal{F}^{n}_{(2(j-1)+1)b_n}\right),
\end{aligned}
$$

which is $o_P\left(1\right)$ given part a4) of Lemma B.1.

Proof of Lemma B.1. Part a6). Here, the proof contains two steps. Step 1. We show show that $\frac{1}{\sqrt{n}}E\left(\sum_{j=1}^{J_n}\sqrt{E\left(\beta\left(p\right)^{2}_{(j-1)(p+1)k_n}|\mathcal{F}^{n}_{(j-1)(p+1)k_n}\right)}\right)\to 0$. Step 2. We show show that $\frac{1}{n}Var\left(\sum_{j=1}^{J_n}\sqrt{E\left(\beta\left(p\right)^{2}_{(j-1)(p+1)k_n}|\mathcal{F}^{n}_{(j-1)(p+1)k_n}\right)}\right)\to 0$. Note that using the first expression in equation (5.47) of Jacod et al. (2009), the result of step 1 follows directly. Given this result, to show step 2, it is sufficient to show that $\frac{1}{n}E\left(\sum_{j=1}^{J_n}\sqrt{E\left(\beta\left(p\right)^{2}_{(j-1)(p+1)k_n}|\mathcal{F}^{n}_{(j-1)(p+1)k_n}\right)}\right)^{2}\to 0.$

We have that

$$\frac{1}{n}\left(\sum_{j=1}^{J_n}\sqrt{E\left(\beta\left(p\right)^2_{(j-1)(p+1)k_n}|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)}\right)^2 \leq \frac{J_n}{n}\sum_{j=1}^{J_n}E\left(E\left(\beta\left(p\right)^2_{(j-1)(p+1)k_n}|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)\right)$$

$$= \frac{J_n}{n}\sum_{j=1}^{J_n}E\left(\beta\left(p\right)^2_{(j-1)(p+1)k_n}\right) \leq K\frac{1}{\sqrt{n}}E\left(\sum_{j=1}^{J_n}\beta\left(p\right)^2_{(j-1)(p+1)k_n}\right),$$

which is $o\left(1\right)$ given equation (5.47) of Jacod et al. (2009) and the fact that $J_n = O\left(\sqrt{n}\right)$ under our assumptions.

Proof of Lemma B.1. Part a7). The proof follows similarly as part a5) and therefore we omit the details.

Our next result is crucial to the proofs of Lemmas 3.1 and 3.2.

Lemma B.2 *Under Assumptions 1, 2 and 3, if $b_n = (p+1)k_n$ where $p \geq 1$ is fixed, then*

a1) $\frac{\sqrt{n}b_n^2}{k_n^2\psi_2^2}\sum_{j=1}^{J_n}\bar{B}_j^2 \to^P V_p + \theta\left(p+1\right)\int_0^1\left(\sigma_s^2 + \frac{\psi_1}{\theta^2\psi_2}\alpha_s\right)^2 ds.$

a2) $\frac{\sqrt{n}b_n^2}{k_n^2\psi_2^2}\sum_{j=1}^{J_n-1}\bar{B}_j\bar{B}_{j+1} \to^P \theta\left(p+1\right)\int_0^1\left(\sigma_s^2 + \frac{\psi_1}{\theta^2\psi_2}\alpha_s\right)^2 ds + O_P\left(\frac{1}{p}\right).$

Proof of Lemma B.2. Part a1). Given the definition of $\bar{B}_j$, we have that

$$\bar{B}_j = \frac{1}{b_n}\sum_{i=(j-1)b_n}^{jb_n-1}\bar{Y}_i^2 = \frac{1}{b_n}\sum_{i=(j-1)b_n}^{jb_n-1}\underbrace{\left(\bar{Y}_i^2 - A_i - c_i\right)}_{\equiv\widetilde{Y}_i} + \frac{1}{b_n}\sum_{i=(j-1)b_n}^{jb_n-1}\left(A_i + c_i\right)$$

where $A_i \equiv E\left(\bar{\epsilon}_i^2|X\right)$ and $c_i = \sum_{j=1}^{k_n} g\left(\frac{j}{k_n}\right)^2\int_{\frac{i+j-1}{n}}^{\frac{i+j}{n}}\sigma_t^2 dt$. It follows that

$$\frac{\sqrt{n}b_n^2}{k_n^2\psi_2^2}\sum_{j=1}^{J_n}\bar{B}_j^2 = \mathcal{B}_{1n} + \mathcal{B}_{2n} + \mathcal{B}_{3n},$$

where

$$\mathcal{B}_{1n} \equiv \sqrt{n}\sum_{j=1}^{J_n}\left(\frac{1}{\theta\psi_2\sqrt{n}}\sum_{i=(j-1)b_n}^{jb_n-1}\widetilde{Y}_i\right)^2 = \sqrt{n}\sum_{j=1}^{J_n}\eta\left(p\right)_j^2,$$

$$\mathcal{B}_{2n} \equiv \frac{2}{\theta\psi_2}\sum_{j=1}^{J_n}\eta\left(p\right)_j\sum_{i=(j-1)b_n}^{jb_n-1}\left(A_i + c_i\right);\ \text{and}$$

$$\mathcal{B}_{3n} \equiv \frac{1}{\theta^2\psi_2^2\sqrt{n}}\sum_{j=1}^{J_n}\left(\sum_{i=(j-1)b_n}^{jb_n-1}\left(A_i + c_i\right)\right)^2.$$

We show that (1) $\mathcal{B}_{1n} \to^P \int_0^1\gamma_t^2\left(p\right)dt$; (2) $\mathcal{B}_{2n} \to^P 0$, and that (3) $\mathcal{B}_{3n} \to^P \left(p+1\right)\theta\int_0^1\left(\sigma_t^2 + \frac{\psi_1}{\theta^2\psi_2}\alpha_t\right)^2 dt$.

Starting with (1), write

$$
\begin{aligned}
\sqrt{n}\sum_{j=1}^{J_n}\eta\left(p\right)_j^2-\int_0^1\gamma_t^2\left(p\right)dt &= \mathcal{B}_{1.1n}+\mathcal{B}_{1.2n}+\mathcal{B}_{1.3n}, \quad \text{with} \\
\mathcal{B}_{1.1n} &= \sqrt{n}\sum_{j=1}^{J_n}\left(\eta\left(p\right)_j^2-E\left(\eta\left(p\right)_j^2|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)\right), \\
\mathcal{B}_{1.2n} &= \sqrt{n}\sum_{j=1}^{J_n}E\left(\eta\left(p\right)_j^2|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)-\frac{N_n}{n}\frac{1}{J_n}\sum_{j=1}^{J_n}\gamma(p)^2_{\frac{j-1}{J_n}}, \\
\mathcal{B}_{1.3n} &= \frac{N_n}{n}\frac{1}{J_n}\sum_{j=1}^{J_n}\gamma(p)^2_{\frac{j-1}{J_n}}-\int_0^1\gamma_t^2\left(p\right)dt.
\end{aligned}
$$

We show that each of $\mathcal{B}_{1.\ell n}\to^P 0$ for $\ell=1,2,3$. For $\ell=1$, by Lenglart's inequality (see e.g. Lemma 4.4 of Vetter (2008)), it is sufficient to show that $n\sum_{j=1}^{J_n}E\left(\eta\left(p\right)_j^4|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)\to^P 0$, which follows immediately by using equation (5.57) of Jacod et al. (2009). Next, to show that $\mathcal{B}_{1.2n}\to^P 0$, note that

$$
\begin{aligned}
\mathcal{B}_{1.2n} &\leq \sum_{j=1}^{J_n}\left|\sqrt{n}E\left(\eta\left(p\right)_j^2|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)-\frac{N_n}{n}\frac{1}{J_n}\gamma(p)^2_{\frac{j-1}{J_n}}\right| \\
&= \sum_{j=1}^{J_n}\left|\sqrt{n}E\left(\frac{1}{\theta^2\psi_2^2 n}\zeta^2(p)_{(j-1)(p+1)k_n}|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)-\frac{1}{n}\left(p+1\right)\theta\sqrt{n}\gamma(p)^2_{\frac{j-1}{J_n}}\right| \\
&= \frac{\sqrt{n}}{\theta^2\psi_2^2 n}\sum_{j=1}^{J_n}\left|E\left(\zeta^2(p)_{(j-1)(p+1)k_n}|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)-\theta^3\psi_2^2\left(p+1\right)\gamma(p)^2_{\frac{j-1}{J_n}}\right| \\
&\leq \frac{K}{\theta^2\psi_2^2\sqrt{n}}\sum_{j=1}^{J_n}\chi(p)_{(j-1)(p+1)k_n}
\end{aligned}
$$

where we use the fact that $N_n/J_n=\left(p+1\right)k_n$ with $k_n=\theta\sqrt{n}$ and rely on equation (5.41) of Jacod et al. (2009) to bound the term in absolute value, where

$$
\chi(p)_{(j-1)(p+1)k_n}=n^{-1/4}+\sqrt{E\left(\beta\left(p\right)^2_{(j-1)(p+1)k_n}|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)}
$$

and $\beta\left(p\right)_i$ is as defined in (9). It follows that

$$
\frac{1}{\sqrt{n}}\sum_{j=1}^{J_n}\chi(p)_{(j-1)(p+1)k_n}\leq\frac{1}{\sqrt{n}}\sum_{j=1}^{J_n}n^{-1/4}+\frac{1}{\sqrt{n}}\sum_{j=1}^{J_n}\sqrt{E\left(\beta\left(p\right)^2_{(j-1)(p+1)k_n}|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)}\to^P 0,
$$

where the first term is of order $O\left(n^{-1/4}\right)$ and the second term is $o_P\left(1\right)$ given part a6) of Lemma B.1. Finally, $\mathcal{B}_{1.3n}\to^P 0$ follows immediately by Riemann's integrability of σ, the fact that $\frac{N_n}{n}\to 1$ and $J_n\to\infty$ as $n\to\infty$.

To show (2), let $\varphi_j \equiv \sum_{i=(j-1)b_n}^{jb_n-1} (A_i + c_i)$ and $\zeta(X,p)_j = \sum_{i=(j-1)b_n}^{jb_n-1} \left(\bar{X}_i^2 - c_i\right)$. We can write

$$\begin{aligned}
\mathcal{B}_{2n} &= \frac{2}{\theta\psi_2}\sum_{j=1}^{J_n} \varphi_j \cdot \eta(p)_j = \mathcal{B}_{2.1n} + \mathcal{B}_{2.2n}, \quad \text{with} \\
\mathcal{B}_{2.1n} &= \frac{2}{\theta\psi_2}\sum_{j=1}^{J_n} \left(\varphi_j \eta(p)_j - E\left(\varphi_j \eta(p)_j | \mathcal{F}^n_{(j-1)(p+1)k_n}\right)\right), \quad \text{and} \\
\mathcal{B}_{2.2n} &= \frac{2}{\theta\psi_2}\sum_{j=1}^{J_n} E\left(\varphi_j \eta(p)_j | \mathcal{F}^n_{(j-1)(p+1)k_n}\right).
\end{aligned}$$

We show that each of $\mathcal{B}_{2.\ell n} \to^P 0$ for $\ell = 1,2$. Note that given the definitions of A_i, c_i, and the fact that $k_n = \theta\sqrt{n}$, Assumption 3 implies that $A_i + c_i \leq K/\sqrt{n}$ uniformly in i. Given that $b_n = (p+1)k_n$, it follows that $\varphi_j \leq K$ uniformly in j. Starting with $\ell = 1$, by Lenglart's inequality, it is sufficient to show that $\sum_{j=1}^{J_n} E\left(\varphi_j^2 \eta(p)_j^2 | \mathcal{F}^n_{(j-1)(p+1)k_n}\right) \to^P 0$. We can write

$$\begin{aligned}
\sum_{j=1}^{J_n} E\left(\varphi_j^2 \eta(p)_j^2 | \mathcal{F}^n_{(j-1)(p+1)k_n}\right) &\leq K\sum_{j=1}^{J_n} E\left(\eta(p)_j^2 | \mathcal{F}^n_{(j-1)(p+1)k_n}\right) \\
&= K\left(\frac{1}{\sqrt{n}}\left(\sqrt{n}\sum_{j=1}^{J_n} E\left(\eta(p)_j^2 | \mathcal{F}^n_{(j-1)(p+1)k_n}\right) - \frac{N_n}{n}\frac{1}{J_n}\sum_{j=1}^{J_n}\gamma(p)^2_{\frac{j-1}{J_n}}\right)\right) \\
&\quad + K\left(\frac{1}{\sqrt{n}}\left(\frac{N_n}{n}\frac{1}{J_n}\sum_{j=1}^{J_n}\gamma(p)^2_{\frac{j-1}{J_n}} - \int_0^1 \gamma_t^2(p)\,dt\right) + \frac{1}{\sqrt{n}}\int_0^1 \gamma_t^2(p)\,dt\right) \\
&\equiv K\left(\frac{1}{\sqrt{n}}\mathcal{B}_{1.2n} + \frac{1}{\sqrt{n}}\mathcal{B}_{1.3n} + \frac{1}{\sqrt{n}}\int_0^1 \gamma_t^2(p)\,dt\right) \\
&= \frac{1}{\sqrt{n}}o_P(1) + \frac{1}{\sqrt{n}}o_P(1) + O_P\left(\frac{1}{\sqrt{n}}\right) = o_P(1),
\end{aligned}$$

where in particular we use the fact that $\mathcal{B}_{1.2n} = o_P(1)$ and $\mathcal{B}_{1.3n} = o_P(1)$, and $\int_0^1 \gamma_t^2(p)\,dt = O_P(1)$. It follows that $\mathcal{B}_{2.1n} \to^P 0$. Next, to show that $\mathcal{B}_{2.2n} \to^P 0$, note that we can write

$$\mathcal{B}_{2.2n} \leq \frac{2K}{\theta\psi_2}\frac{1}{n^{1/4}}\left(n^{1/4}\sum_{j=1}^{J_n} E\left(\eta(p)_j | \mathcal{F}^n_{(j-1)(p+1)k_n}\right)\right) = O_P\left(n^{-1/4}\right) o_P(1) = o_P(1),$$

given that $\varphi_j \leq K$, and given equation (5.49) of Jacod et al. (2009).

Finally, to show (3), note that given the definitions of A_i and c_i, and by using equations (5.23) and (5.36) of Jacod et al. (2009), we can write

$$\sum_{i=(j-1)b_n}^{jb_n-1} (A_i + c_i) = \sum_{i=(j-1)b_n}^{jb_n-1} \left(\frac{\psi_1}{\theta\sqrt{n}}\alpha_{(j-1)b_n/n} + \frac{\theta\psi_2}{\sqrt{n}}\sigma^2_{(j-1)b_n/n}\right) + O\left(\frac{p}{\sqrt{n}} + p\beta(p)_{(j-1)b_n}\right). \tag{11}$$

It follows that

$$\mathcal{B}_{3n} \equiv \frac{1}{\theta^2\psi_2^2\sqrt{n}}\sum_{j=1}^{J_n}\left(\sum_{i=(j-1)b_n}^{jb_n-1}(A_i+c_i)\right)^2 = L_n + R_n,$$

where the leading term is

$$L_n = (p+1)\,\theta\frac{N_n}{n}\frac{1}{J_n}\sum_{j=1}^{J_n}\left(\frac{\psi_1}{\theta^2\psi_2}\alpha_{(j-1)b_n/n}+\sigma^2_{(j-1)b_n/n}\right)^2 \to^P (p+1)\theta\int_0^1\left(\sigma_t^2+\frac{\psi_1}{\theta^2\psi_2}\alpha_t\right)^2 dt. \tag{12}$$

The remainder is such that

$$R_n = K\cdot O_P\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}\sum_{j=1}^{J_n}\beta(p)^2_{(j-1)b_n}\right)\to^P 0$$

by using Lemma (5.4) of Jacod et al. (2009).

Proof of part a2). Recall that $\bar{B}_j = \frac{1}{b_n}\sum_{i=(j-1)b_n}^{jb_n-1}\bar{Y}_i^2$ is the average of observations in a block of size b_n starting at observation $(j-1)\,b_n$. For any integer a_n such that $2\le a_n<b_n$, we can decompose

$$\bar{B}_j = \bar{B}_j^{[0,a_n-1]}+\bar{B}_j^{[a_n,b_n-1]},$$

where $\bar{B}_j^{[0,a_n-1]} \equiv \frac{1}{b_n}\sum\limits_{i=(j-1)b_n}^{(j-1)b_n+a_n-1}\bar{Y}_i^2$ and $\bar{B}_j^{[a_n,b_n-1]} \equiv \frac{1}{b_n}\sum\limits_{i=(j-1)b_n+a_n}^{jb_n-1}\bar{Y}_i^2$. Then

$$\begin{aligned}
\bar{B}_j\bar{B}_{j+1} &= \left(\bar{B}_j^{[0,pk_n-1]}+\bar{B}_j^{[pk_n,b_n-1]}\right)\left(\bar{B}_{j+1}^{[0,k_n-1]}+\bar{B}_{j+1}^{[k_n,b_n-1]}\right)\\
&= \left(\bar{B}_j^{[0,pk_n-1]}\bar{B}_{j+1}^{[0,k_n-1]}\right)+\left(\bar{B}_j^{[0,pk_n-1]}\bar{B}_{j+1}^{[k_n,b_n-1]}\right)+\left(\bar{B}_j^{[pk_n,b_n-1]}\bar{B}_{j+1}^{[k_n,b_n-1]}\right)\\
&+ \left(\bar{B}_j^{[pk_n,b_n-1]}\bar{B}_{j+1}^{[0,k_n-1]}\right)\\
&\equiv \underbrace{\Xi_{1j}+\Xi_{2j}+\Xi_{3j}}_{\equiv L_j}+\Xi_{4j}.
\end{aligned} \tag{13}$$

We can write

$$\frac{n^{1/2}b_n^2}{\psi_2^2k_n^2}\sum_{j=1}^{J_n-1}\bar{B}_j\bar{B}_{j+1} = \frac{n^{1/2}b_n^2}{\psi_2^2k_n^2}\sum_{j=1}^{J_n-1}L_j+\frac{n^{1/2}b_n^2}{\psi_2^2k_n^2}\sum_{j=1}^{J_n-1}\Xi_{4j}.$$

The proof contains two steps. Step 1. We show that $\frac{n^{1/2}b_n^2}{\psi_2^2k_n^2}\sum\limits_{j=1}^{J_n-1}L_j \to^P (p+1)\theta\int_0^1\left(\sigma_t^2+\frac{\psi_1}{\theta^2\psi_2}\alpha_t\right)^2 dt$. Step 2. We show that $\frac{n^{1/2}b_n^2}{\psi_2^2k_n^2}\sum\limits_{j=1}^{J_n-1}\Xi_{4j} = O_P\left(\frac{k_n}{b_n}\right)$.

Step 1. Let $\overline{\sigma\epsilon}_j \equiv \frac{\psi_2 k_n}{n}\sigma^2_{(j-1)b_n/n}+\frac{\psi_1}{k_n}\alpha_{(j-1)b_n/n}$. It follows that

$$\frac{n^{1/2}b_n^2}{\psi_2^2k_n^2}\sum_{j=1}^{J_n-1}L_j-\left((p+1)-\frac{1}{p+1}\right)\theta\int_0^1\left(\sigma_t^2+\frac{\psi_1}{\theta^2\psi_2}\alpha_t\right)^2 dt = \mathcal{B}_{a.1n}+\mathcal{B}_{a.2n}+\mathcal{B}_{a.3n},\ \text{ with}$$

$$\mathcal{B}_{a.1n} = \frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n-1}\left(L_j - E\left(L_j|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)\right),$$

$$\mathcal{B}_{a.2n} = \frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n-1} E\left(L_j|\mathcal{F}^n_{(j-1)(p+1)k_n}\right) - \frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n}\overline{\sigma\epsilon}_j^2,$$

$$\mathcal{B}_{a.3n} = \frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n}\overline{\sigma\epsilon}_j^2 - \left((p+1) - \frac{1}{p+1}\right)\theta\int_0^1\left(\sigma_t^2 + \frac{\psi_1}{\theta^2\psi_2}\alpha_t\right)^2 dt.$$

We show that each of $\mathcal{B}_{a.\ell n} \to^P 0$ for $\ell = 1,2,3$. Starting with $\ell = 1$, by Lenglart's inequality, it is sufficient to show that $\frac{nb_n^4}{\psi_2^4 k_n^4}\sum_{j=1}^{J_n-1} E\left(L_j^2|\mathcal{F}^n_{(j-1)(p+1)k_n}\right) \to^P 0$. We can write

$$\begin{aligned}
\frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n-1} E\left(L_j^2|\mathcal{F}^n_{(j-1)(p+1)k_n}\right) &= \frac{nb_n^4}{\psi_2^4 k_n^4}\sum_{j=1}^{J_n-1} E\left(\left(\bar{B}_j\bar{B}_{j+1} - \Xi_{4j}\right)^2|\mathcal{F}^n_{(j-1)(p+1)k_n}\right) \\
&\leq \frac{nb_n^4}{\psi_2^4 k_n^4}\sum_{j=1}^{J_n-1} E\left(\bar{B}_j^2\bar{B}_{j+1}^2|\mathcal{F}^n_{(j-1)(p+1)k_n}\right) \\
&\leq \frac{nb_n^4}{\psi_2^4 k_n^4}\sum_{j=1}^{J_n-1}\left(E\left(\bar{B}_j^4|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)\right)^{1/2}\left(E\left(\bar{B}_{j+1}^4|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)\right)^{1/2} \\
&= O_P\left(n^{-1/2}\right) = o_P(1),
\end{aligned}$$

where the first line uses the definition of L_j; the second line follows by the fact that $\Xi_{4j} \geq 0$; the third line follows by Cauchy-Schwartz inequality and the fourth line uses the fact that $b_n = (p+1)k_n$, $J_n = O(\sqrt{n})$ and $E\left(\bar{B}_j^4|\mathcal{F}^n_{(j-1)(p+1)k_n}\right) = O_P(n^{-2})$ uniformly in j. To show that $E\left(\bar{B}_j^4|\mathcal{F}^n_{(j-1)(p+1)k_n}\right) = O_P(n^{-2})$, by the c-r inequality,

$$\begin{aligned}
E\left(\bar{B}_j^4|\mathcal{F}^n_{(j-1)(p+1)k_n}\right) &= \frac{1}{b_n^4}E\left(\left(\sum_{i=(j-1)b_n}^{jb_n}\bar{Y}_i^2\right)^4|\mathcal{F}^n_{(j-1)(p+1)k_n}\right) \leq \frac{1}{b_n}E\left(\left(\sum_{i=(j-1)b_n}^{jb_n}\bar{Y}_i^8\right)|\mathcal{F}^n_{(j-1)(p+1)k_n}\right) \\
&\leq Kn^{-2},
\end{aligned}$$

where we can show that

$$E\left(\bar{Y}_i^8\right) \leq K\left(E\left(\bar{X}_i^8\right) + E\left(\bar{\epsilon}_i^8\right)\right) = O\left(n^{-2}\right) \text{ uniformly in i.}$$

given that $\bar{Y}_i = \bar{X}_i + \bar{\epsilon}_i$ and given equations (5.28) and (5.38) of Jacod et al. (2009). This shows that $\mathcal{B}_{a.1n} \to^P 0$. Next, to show that $\mathcal{B}_{a.2n} \to^P 0$, note that given the definition of L_j, the fact that $b_n = (p+1)\, k_n$, and by using equation (13)we can write

$$\begin{aligned}
\frac{n^{1/2} b_n^2}{\psi_2^2 k_n^2} \sum_{j=1}^{J_n-1} E\left(L_j | \mathcal{F}^n_{(j-1)(p+1)k_n}\right) &= n^{1/2} \frac{(p+1)^2}{\psi_2^2} \sum_{j=1}^{J_n-1} E\left(\left(\bar{B}_j^{[0,pk_n-1]} \bar{B}_{j+1}^{[0,k_n-1]}\right) | \mathcal{F}^n_{(j-1)(p+1)k_n}\right) \\
&\quad + n^{1/2} \frac{(p+1)^2}{\psi_2^2} \sum_{j=1}^{J_n-1} E\left(\left(\bar{B}_j^{[0,pk_n-1]} \bar{B}_{j+1}^{[k_n,b_n-1]}\right) | \mathcal{F}^n_{(j-1)(p+1)k_n}\right) \\
&\quad + n^{1/2} \frac{(p+1)^2}{\psi_2^2} \sum_{j=1}^{J_n-1} E\left(\left(\bar{B}_j^{[pk_n,b_n-1]} \bar{B}_{j+1}^{[k_n,b_n-1]}\right) | \mathcal{F}^n_{(j-1)(p+1)k_n}\right) \\
&\equiv \Upsilon_1 + \Upsilon_2 + \Upsilon_3 .
\end{aligned}$$

For Υ_1, we obtain

$$\begin{aligned}
\Upsilon_1 &= n^{1/2} \frac{(p+1)^2}{\psi_2^2} \sum_{j=1}^{J_n-1} E\left(\bar{B}_j^{[0,pk_n-1]} | \mathcal{F}^n_{(j-1)(p+1)k_n}\right) E\left(\bar{B}_{j+1}^{[0,k_n-1]} | \mathcal{F}^n_{(j-1)(p+1)k_n}\right) \\
&= n^{1/2} \frac{(p+1)^2}{\psi_2^2 b_n^2} \sum_{j=1}^{J_n-1} E\left(\left(\sum_{i=(j-1)b_n}^{(j-1)b_n+pk_n-1} \bar{Y}_i^2\right) | \mathcal{F}^n_{(j-1)(p+1)k_n}\right) E\left(\left(\sum_{i=jb_n}^{jb_n+k_n-1} \bar{Y}_i^2\right) | \mathcal{F}^n_{(j-1)(p+1)k_n}\right).
\end{aligned}$$

where we used the fact that $\bar{Y}_i$ and $\bar{Y}_j$ are (conditionally) independent provided that $|j-i| > k_n$. By adding and substracting appropriately, we can write $\bar{Y}_i^2 = \left(\bar{Y}_i^2 - c_i - A_i\right) + (c_i + A_i)$. Then we show that

$$\begin{aligned}
E\left(\sum_{i=(j-1)b_n}^{(j-1)b_n+pk_n-1} \left(\bar{Y}_i^2 - c_i - A_i\right) | \mathcal{F}^n_{(j-1)(p+1)k_n}\right) &= pk_n \varpi_j \text{ and} \qquad (14) \\
E\left(\sum_{i=(j-1)b_n}^{(j-1)b_n+pk_n-1} (c_i + A_i) | \mathcal{F}^n_{(j-1)(p+1)k_n}\right) &= pk_n (\overline{\sigma\epsilon}_j + \varpi_j), \qquad (15)
\end{aligned}$$

where $\varpi_j \equiv O_P\left(\frac{1}{n} + \frac{1}{\sqrt{n}} \sqrt{E\left(\beta\,(2p+1)^2_{(j-1)(p+1)k_n} | \mathcal{F}^n_{(j-1)(p+1)k_n}\right)}\right)$ and $\overline{\sigma\epsilon}_j \equiv \frac{\psi_2 k_n}{n} \sigma^2_{(j-1)b_n/n} + \frac{\psi_1}{k_n} \alpha_{(j-1)b_n/n}$. Given the decomposition $\bar{Y}_i^2 = \left(\bar{X}_i + \bar{\epsilon}_i\right)^2 = \bar{X}_i^2 + 2\bar{X}_i \bar{\epsilon}_i + \bar{\epsilon}_i^2$, we can write

$$\begin{aligned}
E\left(\sum_{i=(j-1)b_n}^{(j-1)b_n+pk_n-1} \left(\bar{Y}_i^2 - c_i - A_i\right) | \mathcal{F}^n_{(j-1)(p+1)k_n}\right) &= E\left(\sum_{i=(j-1)b_n}^{(j-1)b_n+pk_n-1} \left(\bar{X}_i^2 - c_i\right) | \mathcal{F}^n_{(j-1)(p+1)k_n}\right) \\
&\quad + E\left(\sum_{i=(j-1)b_n}^{(j-1)b_n+pk_n-1} \left(2\bar{X}_i \bar{\epsilon}_i + \bar{\epsilon}_i^2 - A_i\right) | \mathcal{F}^n_{(j-1)(p+1)k_n}\right) \\
&\equiv \zeta_1(X) + \zeta_1(X, \epsilon) .
\end{aligned}$$

We can show that $\zeta_1(X,\epsilon) = 0$ by relying on Assumption 1. In particular, noting that $\mathcal{F}^n_{(j-1)(p+1)k_n} \subset \mathcal{F}^0 \times \mathcal{F}^1_{\frac{(j-1)(p+1)k_n}{n}-}$, (where $\mathcal{F}^0 \times \mathcal{F}^1_{\frac{(j-1)(p+1)k_n}{n}-}$ denotes the sigma algebra generated by all $\mathcal{F}^0$-measurable random variables plus all variables Y_s, with $s < (j-1)\,(p+1)\,k_n$) we have that by the law of iterated expectations,

$$\zeta_1(X,\epsilon)=E\left(E\left(\sum_{i=(j-1)b_n}^{(j-1)b_n+pk_n-1}\left(2\bar{X}_i\bar{\epsilon}_i+\bar{\epsilon}_i^2-A_i\right)|\mathcal{F}^0\times\mathcal{F}^1_{\frac{(j-1)(p+1)k_n}{n}-}\right)|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)=0,$$

where by Assumption 1, $E\left(\bar{X}_i\bar{\epsilon}_i|\mathcal{F}^0\times\mathcal{F}^1_{\frac{(j-1)(p+1)k_n}{n}-}\right)=\bar{X}_iE\left(\bar{\epsilon}_i|\mathcal{F}^0\times\mathcal{F}^1_{\frac{(j-1)(p+1)k_n}{n}-}\right)=\bar{X}_iE\left(\bar{\epsilon}_i|X\right)=0$ and $E\left(\bar{\epsilon}_i^2|\mathcal{F}^0\times\mathcal{F}^1_{\frac{(j-1)(p+1)k_n}{n}-}\right)=E\left(\bar{\epsilon}_i^2|X\right)\equiv A_i$ (see equation (5.37) of Jacod et al. (2009)). For $\zeta_1(X)$, by the definition of c_i, we can write

$$\begin{aligned}
\zeta_1(X) &\leq \frac{K}{n^{1/4}}\chi(p)_{(j-1)(p+1)k_n}\\
&= K\left(\frac{1}{n^{1/2}}+\frac{1}{n^{1/4}}\sqrt{E\left(\beta\left(p\right)^2_{(j-1)(p+1)k_n}|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)}\right)\\
&\leq \left(\frac{1}{n^{1/2}}+\frac{1}{n^{1/4}}\sqrt{E\left(\beta\left(p\right)^2_{(j-1)(p+1)k_n}|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)}\right)\\
&= pk_n\varpi_j,
\end{aligned}$$

where the first line follows from equation (5.30) of Jacod et al. (2009); the second line uses the definition of $\chi(p)_{(j-1)(p+1)k_n}=n^{-1/4}+\sqrt{E\left(\beta\left(p\right)^2_{(j-1)(p+1)k_n}|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)}$; and the third line uses the fact that $\beta\left(p\right)_{(j-1)(p+1)k_n}\leq\beta\left(2p+1\right)_{(j-1)(p+1)k_n}$. This proves (14). To show (15), we rely on arguments similar to those used by Jacod et al. (2009) (in particular, see their equations (5.23) and (5.36)). This implies that

$$E\left(\left(\sum_{i=(j-1)b_n}^{(j-1)b_n+pk_n-1}\bar{Y}_i^2\right)|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)=pk_n\left(\overline{\sigma\epsilon}_j+2\varpi_j\right).$$

By similar arguments, we can show that

$$E\left(\left(\sum_{i=jb_n}^{jb_n+k_n-1}\bar{Y}_i^2\right)|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)=k_n\left(\overline{\sigma\epsilon}_j+2\varpi_j\right),$$

which implies that

$$\begin{aligned}
\Upsilon_1 &= n^{1/2}\frac{(p+1)^2}{\psi_2^2b_n^2}\sum_{j=1}^{J_n-1}pk_n^2\left(\overline{\sigma\epsilon}_j+2\varpi_j\right)^2=n^{1/2}\frac{(p+1)^2}{\psi_2^2b_n^2}pk_n^2\sum_{j=1}^{J_n-1}\left(\overline{\sigma\epsilon}_j^2+4\varpi_j\overline{\sigma\epsilon}_j+2\varpi_j^2\right)\\
&= n^{1/2}\frac{p}{\psi_2^2}\sum_{j=1}^{J_n-1}\left(\overline{\sigma\epsilon}_j^2+4\varpi_j\overline{\sigma\epsilon}_j+2\varpi_j^2\right). \qquad (16)
\end{aligned}$$

Using the fact that $\sqrt{n}\overline{\sigma\epsilon}_j=O_P\left(1\right)$ uniformly in j, $J_n=O\left(\sqrt{n}\right)$, and the definition of $\varpi_j=$

$O_P\left(\frac{1}{n}+\frac{1}{\sqrt{n}}E\left(\left(\beta\left(2p+1\right)_{(j-1)(p+1)k_n}\right)|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)\right)$ uniformly in j, we get that

$$\begin{aligned} n^{1/2}\frac{2p}{\psi_2^2}\sum_{j=1}^{J_n-1}\varpi_j\overline{\sigma\epsilon}_j &= \frac{2p}{\psi_2^2}\sum_{j=1}^{J_n-1}\varpi_j\left(n^{1/2}\overline{\sigma\epsilon}_j\right) \\ &= O_P\left(\frac{J_n}{n}\right)+O_P\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{J_n}\sqrt{E\left(\beta\left(2p+1\right)^2_{(j-1)(p+1)k_n}|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)}\right), \end{aligned}$$

which is $o_P(1)$ since $J_n/n=O\left(1/\sqrt{n}\right)$ and $\frac{1}{\sqrt{n}}\sum_{j=1}^{J_n-1}E\left(\left(\beta\left(2p+1\right)_{(j-1)(p+1)k_n}\right)|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)\to^P 0$ by Lemma B.1. The third term in (16) is such that

$$\begin{aligned} n^{1/2}\frac{2p}{\psi_2^2}\sum_{j=1}^{J_n-1}\varpi_j^2 &= O_P\left(\frac{J_n}{n^{3/2}}\right)+O_P\left(\frac{2}{\sqrt{n}}\frac{1}{\sqrt{n}}\sum_{j=1}^{J_n}\sqrt{E\left(\beta\left(2p+1\right)^2_{(j-1)(p+1)k_n}|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)}\right) \\ &\quad +O_P\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{J_n-1}E\left(\left(\beta\left(2p+1\right)^2_{(j-1)(p+1)k_n}\right)|\mathcal{F}^n_{(j-1)(p+1)k_n}\right)\right)=o_P(1), \end{aligned}$$

given parts a5) and a7) of Lemma B.1. Thus

$$\Upsilon_1=n^{1/2}\frac{p}{\psi_2^2}\sum_{j=1}^{J_n-1}\overline{\sigma\epsilon}_j^2+o_P(1). \tag{17}$$

Similarly, we can show

$$\Upsilon_2 = n^{1/2}\frac{p^2}{\psi_2^2}\sum_{j=1}^{J_n-1}\overline{\sigma\epsilon}_j^2+o_P(1)\text{, and} \tag{18}$$

$$\Upsilon_3 = n^{1/2}\frac{1}{\psi_2^2}\sum_{j=1}^{J_n-1}\overline{\sigma\epsilon}_j^2+o_P(1). \tag{19}$$

From (17), (18) and (19), we have that

$$\begin{aligned} \frac{n^{1/2}b_n^2}{\psi_2^2k_n^2}\sum_{j=1}^{J_n-1}E\left(L_j|\mathcal{F}^n_{(j-1)(p+1)k_n}\right) &= \left((p+1)^2-1\right)\frac{n^{1/2}}{\psi_2^2}\sum_{j=1}^{J_n-1}\overline{\sigma\epsilon}_j^2+o_P(1) \\ &= \left((p+1)^2-1\right)\left(\frac{n^{1/2}}{\psi_2^2}\sum_{j=1}^{J_n}\overline{\sigma\epsilon}_j^2-\frac{n^{1/2}}{\psi_2^2}\overline{\sigma\epsilon}_{J_n}^2\right)+o_P(1) \\ &= \left((p+1)^2-1\right)\left(\frac{n^{1/2}}{\psi_2^2}\sum_{j=1}^{J_n}\overline{\sigma\epsilon}_j^2\right)+O_P\left(n^{-1/2}\right)+o_P(1) \\ &= \left((p+1)^2-1\right)\left(\frac{n^{1/2}}{\psi_2^2}\sum_{j=1}^{J_n}\overline{\sigma\epsilon}_j^2\right)+o_P(1). \end{aligned}$$

This shows that $\mathcal{B}_{a.2n}\to^P 0$. Finally, $\mathcal{B}_{a.3n}\to^P 0$ follows immediately by Riemann's integrability of α and σ, the fact that $\frac{N_n}{n}\to 1$ and $J_n\to\infty$ as $n\to\infty$.

Step 2. Next, we analyze the term that depends on $\Xi_{4j} \equiv \bar{B}_j^{[pk_n, b_n-1]} \bar{B}_{j+1}^{[0,k_n-1]}$. We show that $E\left(\frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n-1}\Xi_{4j}\right) = O\left(\frac{k_n}{b_n}\right)$, and $Var\left(\frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n-1}\Xi_{4j}\right) = O\left(\frac{k_n^2}{b_n^2}\right)$. Given the definition of Ξ_{4j}, we have that

$$\begin{aligned}
E\left(\frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n-1}\Xi_{4j}\right) &= \frac{n^{1/2}}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n-1} E\left(\left(\sum_{i=1}^{k_n}\bar{Y}_{i-1+(j-1)b_n+pk_n}^2\right)\left(\sum_{l=1}^{k_n}\bar{Y}_{l-1+jb_n}^2\right)\right) \\
&= \frac{n^{1/2}}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n-1}\sum_{i=1}^{k_n}\sum_{l=1}^{k_n} E\left(\bar{Y}_{i-1+(j-1)b_n+pk_n}^2 \bar{Y}_{l-1+jb_n}^2\right) \\
&\leq \frac{n^{1/2}}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n-1}\sum_{i=1}^{k_n}\sum_{l=1}^{k_n}\left(E\left(\bar{Y}_{i-1+(j-1)b_n+pk_n}^4\right)\right)^{1/2}\left(E\left(\bar{Y}_{l-1+jb_n}^4\right)\right)^{1/2}. \quad (20)
\end{aligned}$$

Given the decomposition $\bar{Y}_i = \bar{X}_i + \bar{\epsilon}_i$, using the triangle inequality we have that $\left|\bar{Y}_i\right| \leq \left|\bar{X}_i\right| + \left|\bar{\epsilon}_i\right|$. It follows that

$$E\left(\bar{Y}_i^4\right) \leq K\left(E\left(\bar{X}_i^4\right) + E\left(\bar{\epsilon}_i^4\right)\right) = O\left(n^{-1}\right)$$

uniformly in i, where we use the c-r inequality and equations (5.28) and (5.38) of Jacod et al. (2009) to show that $E\left(\bar{X}_i^4\right) = O\left(n^{-1}\right)$, and $E\left(\bar{\epsilon}_i^4\right) = O\left(n^{-1}\right)$ uniformly in i. Thus, we can bound (20) by $K\frac{n^{1/2}}{\psi_2^2 k_n^2}\frac{J_n k_n^2}{n} = O\left(\frac{k_n}{b_n}\right)$ given that $J_n = N_n/b_n = \left(\sqrt{n}/\theta\right)\left(N_n/n\right)\left(k_n/b_n\right)$ with $k_n = \theta\sqrt{n}$ and $N_n/n \to 1$. Next, we show that $Var\left(\frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n-1}\Xi_{4j}\right) = O\left(\frac{k_n^2}{b_n^2}\right)$. We have that $Var\left(\frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n-1}\Xi_{4j}\right) \leq E\left(\frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n-1}\Xi_{4j}\right)^2$. Thus, using the c-r inequality, we can write

$$\begin{aligned}
Var\left(\frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n-1}\Xi_{4j}\right) &\leq \frac{n b_n^4 J_n}{\psi_2^4 k_n^4}\sum_{j=1}^{J_n-1} E\left(\Xi_{4j}^2\right) \\
&= \frac{n J_n}{\psi_2^4 k_n^4}\sum_{j=1}^{J_n-1}\sum_{i=1}^{k_n}\sum_{l=1}^{k_n}\sum_{s=1}^{k_n}\sum_{u=1}^{k_n} E\left(\bar{Y}_{i-1+(j-1)b_n+pk_n}^2 \bar{Y}_{l-1+jb_n}^2 \bar{Y}_{s-1+(j-1)b_n+pk_n}^2 \bar{Y}_{u-1+jb_n}^2\right) \\
&\leq K\frac{n J_n}{\psi_2^4 k_n^4} J_n k_n^4 E\left(\bar{Y}_i^8\right), \quad (21)
\end{aligned}$$

where the second inequality holds given Cauchy-Schwartz inequality and the fact that $E\left(\bar{Y}_i^8\right) = O\left(n^{-2}\right)$ uniformly in i. Thus, we can bound (21) by $K\frac{J_n^2}{n} = O\left(\frac{k_n^2}{b_n^2}\right)$ given that $J_n = N_n/b_n = \left(\sqrt{n}/\theta\right)\left(N_n/n\right)\left(k_n/b_n\right)$ with $k_n = \theta\sqrt{n}$ and $N_n/n \to 1$. Hence $\frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n-1}\Xi_{4j} = O_P\left(\frac{k_n}{b_n}\right)$.

Proof of Lemma 3.1. Part a) Given the definition of V_n^*, we can write

$$V_n^* = V_{1n}^* - \frac{\sqrt{n}N_n b_n}{\left(N_n - b_n + 1\right)^2}V_{2n}^*,$$

where

$$V_{1n}^* = \frac{1}{b_n}\sum_{t=0}^{b_n-1} v_{1n,t}^*, \text{ with } v_{1n,t}^* \equiv \frac{\sqrt{n}}{(N_n - b_n + 1)\, N_n} \sum_{j=1}^{\left[\frac{N_n-t}{b_n}\right]} \left(\sum_{i=t+1}^{b_n+t} Z_{i+(j-1)b_n}\right)^2, \text{ and}$$

$$V_{2n}^* = \frac{1}{b_n}\sum_{t=0}^{b_n-1} v_{2n,t}^*, \text{ with } v_{2n,t}^* \equiv \frac{1}{N_n} \sum_{j=1}^{\left[\frac{N_n-t}{b_n}\right]} \sum_{i=t+1}^{b_n+t} Z_{i+(j-1)b_n}.$$

We now proceed in two steps. In Step 1, we show that $v_{1n,t}^* \to^P V_p + \theta\,(p+1)\int_0^1 \left(\sigma_s^2 + \frac{\psi_1}{\theta^2\psi_2}\alpha_s\right)^2 ds$ uniformly in t. In Step 2, we show that $v_{2n,t}^* \to^P \int_0^1 \left(\sigma_s^2 + \frac{\psi_1}{\theta^2\psi_2}\alpha_s\right)^2 ds$, also uniformly in t. This together with the fact that $\frac{\sqrt{n}N_n b_n}{(N_n-b_n+1)^2} \to (p+1)\,\theta$ as $n \to \infty$ when $b_n = (p+1)\,k_n$ and k_n satisfies Assumption 2 imply the result. Proof of Step 1. For $t = 0, \ldots, b_n - 1$ and $j = 1, \ldots, \left[\frac{N_n-t}{b_n}\right]$, let

$$\bar{B}_{j,t} \equiv \frac{1}{b_n}\sum_{i=1}^{b_n} \bar{Y}_{i-1+t+(j-1)b_n}^2 = \frac{k_n\psi_2}{N_n}\frac{1}{b_n}\sum_{i=1}^{b_n} Z_{i+t+(j-1)b_n},$$

where $Z_i \equiv \frac{N_n}{k_n}\frac{1}{\psi_2}\bar{Y}_{i-1}^2$ and note that the $\bar{B}_{j,t}$ are averages of non-overlapping blocks for given t. With this notation, we have that

$$v_{1n,t}^* = \frac{N_n^2}{(N_n - b_n + 1)\, N_n}\frac{\sqrt{n} b_n^2}{k_n^2\psi_2^2} \sum_{j=1}^{\left[\frac{N_n-t}{b_n}\right]} \bar{B}_{j,t}^2,$$

where we can show that $\frac{N_n^2}{(N_n-b_n+1)N_n} \to 1$ under the condition that $b_n = (p+1)\,k_n$. Using arguments similar to those used to prove Lemma B.2, we can show that

$$\frac{\sqrt{n} b_n^2}{k_n^2\psi_2^2} \sum_{j=1}^{\left[\frac{N_n-t}{b_n}\right]} \bar{B}_{j,t}^2 \to^P V_p + \theta\,(p+1)\int_0^1 \left(\sigma_s^2 + \frac{\psi_1}{\theta^2\psi_2}\alpha_s\right)^2 ds$$

uniformly in t. The proof of Step 2 relies on the consistency result in Theorem 1 of Christensen, Kinnebrock and Podolskij (2010). Indeed $v_{2n,t}^*$ is the main term in Jacod et al. (2009) pre-averaged realized volatility estimator without the bias corrected term, with starting point t. **Part b).** Follows directly from part a) of Lemma 3.1 when replacing σ_t by a constant for all t. **Part c).** Follows directly from part a) of Lemma 3.1.

Proof of Lemma 3.2 Part a). Given the definition of V_n^*, we can write

$$\begin{aligned} V_n^* &= Var^*\left(n^{1/4}PRV_n^*\right) = \frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\sum_{j=1}^{J_n-1}\left(\bar{B}_j - \bar{B}_{j+1}\right)^2 Var^*\left(\eta_j^2\right) \\ &= 2Var^*(\eta)\,\frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\left(\sum_{j=1}^{J_n}\bar{B}_j^2 - \sum_{j=1}^{J_n-1}\bar{B}_j\bar{B}_{j+1}\right) \\ &\quad - Var^*(\eta)\,\frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\left(\bar{B}_1^2 + \bar{B}_{J_n}^2\right). \end{aligned}$$

The result follows from Lemma B.2 and the fact that $\frac{n^{1/2}b_n^2}{\psi_2^2 k_n^2}\left(\bar{B}_1^2+\bar{B}_{J_n}^2\right)=O_P\left(\frac{1}{\sqrt{n}}\right)$ given that $\bar{B}_j=O_P\left(1/\sqrt{n}\right)$ uniformly in j. **Part b).** Follows directly from Lemma 3.2.a) and the assumptions that $Var^*\left(\eta\right)=\frac{1}{2}$ that $p\to\infty$.

Proof of Theorem 3.1 For any fixed $p\geq 1$, let $S_n^*=n^{1/4}\left(PRV_n^*-E^*\left(PRV_n^*\right)\right)=\frac{b_n}{\psi_2 k_n}\sum_{j=1}^{J_n} z_j^*$, where $z_j^*=n^{1/4}\frac{b_n}{\psi_2 k_n}\left(\bar{B}_j{}^*-E^*\left(\bar{B}_j{}^*\right)\right)$. It follows that $E^*\left(\sum_{j=1}^{J_n} z_j^*\right)=0$, and

$$V_n^*\equiv Var^*\left(\sum_{j=1}^{J_n} z_j^*\right)\xrightarrow{P} V_p+O_P\left(\frac{1}{p}\right)\equiv\tilde{V}_p.$$

Since $z_1^*,\cdots,z_{J_n}^*$ are conditionally independent, by the Berry-Esseen bound, for some small $\delta>0$ and for some constant $C_p>0$,

$$\sup_{x\in\Re}\left|P^*\left(S_n^*\leq x\right)-\Phi\left(x/\sqrt{\tilde{V}_p}\right)\right|\leq C_p\sum_{j=1}^{J_n}E^*\left|z_j^*\right|^{2+\delta},$$

which converges to zero in probability as $n\to\infty$. We have

$$\begin{aligned}\sum_{j=1}^{J_n}E^*\left|z_j^*\right|^{2+\delta} &= \sum_{j=1}^{J_n}E^*\left|n^{1/4}\frac{b_n}{\psi_2 k_n}\left(\bar{B}_j^*-E^*\left(\bar{B}_j^*\right)\right)\right|^{2+\delta}\\ &\leq 2n^{\frac{(2+\delta)}{4}}\left(\frac{b_n}{\psi_2 k_n}\right)^{2+\delta}\sum_{j=1}^{J_n}E^*\left|\bar{B}_j^*\right|^{2+\delta}\\ &\leq 2K_p n^{\frac{(2+\delta)}{4}}E^*\left|\eta_1\right|^{2+\delta}\sum_{j=1}^{J_n}\left|\bar{B}_j\right|^{2+\delta}=K_p O_p\left(n^{-\frac{\delta}{4}}\right)=o_p\left(1\right),\end{aligned}$$

since $E^*\left|\eta_j\right|^{2+\delta}\leq\Delta<\infty$, $\bar{B}_j=K_p O_p\left(\frac{1}{\sqrt{n}}\right)$, and $J_n\sim n^{1/2}$. It follows that $n^{1/4}\left(PRV_n^*-E^*\left(PRV_n^*\right)\right)\to^{d^*} N(0,\tilde{V}_p)$ in probability, for any fixed $p\geq 1$. The result follows by using part b) of Lemma 3.2 and letting $p\to\infty$.

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