Econometrica, Vol. 77, No. 1 (January, 2009), 283–306

BOOTSTRAPPING REALIZED VOLATILITY

BY SÍLVIA GONÇALVES AND NOUR MEDDAHI[1]

We propose bootstrap methods for a general class of nonlinear transformations of realized volatility which includes the raw version of realized volatility and its logarithmic transformation as special cases. We consider the independent and identically distributed (i.i.d.) bootstrap and the wild bootstrap (WB), and prove their first-order asymptotic validity under general assumptions on the log-price process that allow for drift and leverage effects. We derive Edgeworth expansions in a simpler model that rules out these effects. The i.i.d. bootstrap provides a second-order asymptotic refinement when volatility is constant, but not otherwise. The WB yields a second-order asymptotic refinement under stochastic volatility provided we choose the external random variable used to construct the WB data appropriately. None of these methods provides third-order asymptotic refinements. Both methods improve upon the first-order asymptotic theory in finite samples.

KEYWORDS: Realized volatility, i.i.d. bootstrap, wild bootstrap, Edgeworth expansions.

1. INTRODUCTION

THE INCREASING AVAILABILITY of high frequency financial data has contributed to the popularity of realized volatility as a measure of volatility in finance. Realized volatility is simple to compute (it is equal to the sum of squared high frequency returns) and is a consistent estimator of integrated volatility under general conditions (see Andersen, Bollerslev, and Diebold (2002) for a survey of realized volatility).

Recently, a series of papers, including Barndorff-Nielsen and Shephard (henceforth BNS) (2002) and Barndorff-Nielsen, Graversen, Jacod, and Shephard (BNGJS) (2006) have developed an asymptotic theory for measures of variation such as realized volatility. In particular, for a rather general stochastic volatility model, these authors establish a central limit theorem (CLT) for

[1]We would like to thank participants at the 2005 North American Winter Meeting of the Econometric Society, the SBFSIF II conference, Québec (April 2005), the CIREQ Montréal Financial Econometrics (May 2005), the SETA conference, Taipei (May 2005), the 2005 CEA meetings, the Princeton–Chicago High Frequency Conference (June 2005), and the NBER Summer Institute 2005, as well as seminar participants at Concordia University, Université de Toulouse I, the St. Louis Fed, and Universidade Nova de Lisboa. We also thank Torben Andersen, António Antunes, Christian Brownlees, Rui Castro, Valentina Corradi, Peter Hansen, Emma Iglesias, Atsushi Inoue, Lutz Kilian, and especially Per Mykland and Neil Shephard for helpful comments on the first version of the paper. In addition, we are grateful to three anonymous referees and a co-editor for many valuable suggestions. This work was supported by grants from FQRSC, SSHRC, MITACS, NSERC, and Jean-Marie Dufour's Econometrics Chair of Canada. Parts of this paper were completed while Gonçalves was visiting the Banco de Portugal, Lisboa, and the Finance Department at Stern Business School and Meddahi was visiting Toulouse University and CREST–Paris.

© 2009 The Econometric Society DOI: 10.3982/ECTA5971

realized volatility over a fixed interval of time, for example, a day, as the number of intraday returns increases to infinity.

In this paper, we propose bootstrap methods for realized volatility-like measures. Our main motivation is to improve upon the existing asymptotic mixed normal approximations. The bootstrap can be particularly valuable in the context of high frequency data-based measures. Current practice is to use a moderate number of intraday returns in computing realized volatility to avoid microstructure biases.[2] Sampling at long horizons may limit the value of the asymptotic approximations derived under the assumption of an infinite number of returns. In particular, the Monte Carlo results in BNS (2005) showed that the feasible asymptotic theory for realized volatility can be a poor guide to the finite sample distribution of the studentized realized volatility. BNS (2005) also showed that a logarithmic version of the raw statistic has improved finite sample properties.

Here we focus on a general class of nonlinear transformations of realized volatility which includes the raw realized volatility and its log transform as special cases. For this class of statistics, we ask whether we can improve upon the existing first-order asymptotic theory by relying on the bootstrap for inference on integrated volatility in the absence of microstructure noise. Since the effects of microstructure noise are more pronounced at very high frequencies, we expect the bootstrap to be a useful tool of inference based on realized volatility when sampling at moderate frequencies such as 30 minute horizons (as in Andersen, Bollerslev, Diebold, and Labys (2003)) or at 10–15 minute horizons for liquid asset returns (see Hansen and Lunde (2006)).

We propose and analyze two bootstrap methods for realized volatility: an independent and identically distributed (i.i.d.) bootstrap and a wild bootstrap (WB). The i.i.d. bootstrap generates bootstrap intraday returns by resampling with replacement the original set of intraday returns. It is motivated by a benchmark model in which volatility is constant and therefore intraday returns are i.i.d. In practice, volatility has components which are highly persistent, especially over a daily horizon, implying that it is at least locally nearly constant. Hence we may expect the i.i.d. bootstrap to provide a good approximation even under stochastic volatility. The WB observations are generated by multiplying each original intraday return by an i.i.d. draw from a distribution that is independent of the data. The WB was introduced by Wu (1986), and further studied by Liu (1988) and Mammen (1993) in the context of cross-section linear regression models subject to unconditional heteroskedasticity in the error term.

We summarize our main contributions as follows. First, we prove the first-order asymptotic validity of both bootstrap methods under very general as-

[2]Recently, a number of papers have studied the impact of microstructure noise on realized volatility; these include Zhang, Mykland, and Aït-Sahalia (2005b), Hansen and Lunde (2006), Bandi and Russell (2008), and Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008). In particular, these papers proposed alternative estimators of integrated volatility that are robust to microstructure noise and that differ from realized volatility.

sumptions which allow for drift and leverage effects. Second, for a simpler model that rules out these effects, we derive formal second- and third-order Edgeworth expansions of the distribution of realized volatility-based t statistics as well as of their bootstrap analogues. Third, we use our Edgeworth expansions to compare the accuracy of the first-order asymptotic theory for realized volatility and for its log transform. Last, we use our Edgeworth expansions and Monte Carlo simulations to compare the finite sample accuracy of bootstrap confidence intervals for integrated volatility with the existing CLT-based intervals.

Our results are as follows. The Edgeworth expansions for the raw and log statistics provide a theoretical explanation for the superior finite sample performance of the log statistic. For both types of statistics, the simulated bootstrap (one-sided and two-sided symmetric) intervals are more accurate in finite samples than the CLT-based intervals. The second-order Edgeworth expansions show that the i.i.d. bootstrap provides a second-order refinement over the normal approximation when volatility is constant but not otherwise. When volatility is time-varying and the rate of convergence of both approximations is the same, we use the asymptotic relative bootstrap error as a criterion of comparison (see Shao and Tu (1995) and Davidson and Flachaire (2001) for a similar argument). We show that the i.i.d. bootstrap is better than the normal approximation under this criterion for the raw statistic. These results are consistent with the good finite sample properties of the i.i.d. bootstrap one-sided confidence intervals. The WB provides a second-order asymptotic refinement when we choose the external random variable appropriately. We provide an optimal choice for the raw statistic. Our Monte Carlo simulations show that the WB implemented with this choice outperforms the first-order asymptotic normal approximation. The comparison between this WB and the i.i.d. bootstrap favors the i.i.d. bootstrap, which is the preferred method in the context of our study.

Motivated by the good finite sample performance of the bootstrap for two-sided symmetric intervals, we also investigate the ability of the bootstrap to provide a third-order asymptotic refinement for the raw realized volatility statistic. We show that none of our bootstrap methods gives third-order refinements. This is true for the i.i.d. bootstrap even when volatility is constant, a surprising result given that returns are i.i.d. in this case.

A distinctive feature of our i.i.d. bootstrap t statistic is that it uses the (unscaled) sample variance estimator of the bootstrap squared returns and not the bootstrap analogue of the variance estimator proposed by BNS (2002) (which relies on the conditional local Gaussianity of intraday returns and cannot be used with the bootstrap). Under constant volatility, an alternative consistent variance estimator to BNS (2002) is the (unscaled) sample variance of squared returns, which mimics the i.i.d. bootstrap variance estimator. In this case, the i.i.d. bootstrap is third-order accurate when used to estimate the distribution of the alternative t statistic based on the sample variance of squared returns.

Thus, the lack of third-order asymptotic refinements for the i.i.d. bootstrap under constant volatility is explained by the fact that the bootstrap statistic is not of the same form as the original statistic.

The remainder of this paper is organized as follows. In Section 2, we describe the setup and briefly review the existing theory. Section 3 introduces the bootstrap methods and establishes their first-order asymptotic validity. Section 4 contains the second-order accuracy results, whereas Section 5 discusses third-order results. Section 6 contains simulations and Section 7 concludes. In Appendix A we state and prove the cumulant asymptotic expansions. Appendix B collects some of the proofs of the results that appear in Sections 3–5. Supplementary proofs and technical results appear in the web supplement to this paper (Gonçalves and Meddahi (2009), hereafter GM09).

2. SETUP, NOTATION, AND EXISTING THEORY

We follow BNGJS (2006) and assume that the log-price process $\{\log S_t : t \geq 0\}$ is defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and follows the continuous time process

$$d\log S_t = \mu_t\,dt + \sigma_t\,dW_t, \tag{1}$$

where W_t denotes a standard Brownian motion, μ is an adapted predictable locally bounded drift term, and σ is an adapted cadlag volatility process. These assumptions are very general, allowing for jumps, intraday seasonality, and long memory in both μ and σ. In addition, we do not assume W_t to be independent of σ_t, allowing for the presence of leverage effects. The parameter of interest is the integrated volatility over a fixed time interval $[0, 1]$ and is defined as $\overline{\sigma^2} \equiv \int_0^1 \sigma_u^2\,du$. A consistent estimator of $\overline{\sigma^2}$ is the realized volatility $R_2 = \sum_{i=1}^{1/h} r_i^2$, where $r_i \equiv \log S_{ih} - \log S_{(i-1)h}$ denotes the high frequency return measured over the period $[(i-1)h, ih]$ for $i = 1, \dots, 1/h$.

For any $q > 0$, define $\overline{\sigma^q} \equiv \int_0^1 \sigma_u^q\,du$ and $\overline{\sigma_h^q} \equiv h^{-q/2+1}\sum_{i=1}^{1/h}(\sigma_i^2)^{q/2}$, where $\sigma_i^2 \equiv \int_{(i-1)h}^{ih} \sigma_u^2\,du$. BNGJS (2006) showed that for any $q > 0$, as $h \to 0$, $R_q \equiv h^{-q/2+1}\sum_{i=1}^{1/h}|r_i|^q \xrightarrow{P} \mu_q\overline{\sigma^q}$, where $\mu_q \equiv E|Z|^q$, with $Z \sim N(0,1)$. When $q = 2$, we obtain the consistency result for realized volatility. BNGJS (2006) also showed that

$$T_h \equiv \frac{\sqrt{h^{-1}}(R_2 - \overline{\sigma^2})}{\sqrt{\hat{V}}} \xrightarrow{d} N(0,1), \tag{2}$$

where $\hat{V} = \frac{2}{3}R_4$, under very general conditions, including drift and leverage effects. In particular, a sufficient assumption is (1) and

$$\sigma_t = \sigma_0 + \int_0^t a_u^{\#}\,du + \int_0^t \sigma_u^{\#}\,dW_u + \int_0^t v_u^{\#}\,dV_u, \tag{3}$$

with $a^\#$, $\sigma^\#$, and $v^\#$ adapted cadlag processes, $a^\#$ predictable and locally bounded, and V a Brownian motion independent of W. Equation (3) does not allow for jumps in the volatility, but this can be relaxed (see Assumption H1 of BNGJS (2006) for a more general assumption on σ). An earlier statement of the CLT result for realized volatility under stronger conditions appeared in Jacod and Protter (1998) and BNS (2002).

The log transformation of realized volatility is often used in empirical applications due to its improved finite sample properties. Here we consider a general class of nonlinear transformations that satisfy the following assumption. Throughout we let $g'(z)$ and $g''(z)$ denote the first and second derivatives of g with respect to z, respectively.

ASSUMPTION G: *Let $g:\mathbb{R}\to\mathbb{R}$ be twice continuously differentiable with $g'(\overline{\sigma^2})\neq 0$ for any path of σ.*

Assumption G contains the log transform for realized volatility (when $g(z) = \log z$) and the raw statistic (when $g(z) = z$) as special cases. The corresponding t statistic is

$$T_{g,h} \equiv \frac{\sqrt{h^{-1}}(g(R_2) - g(\overline{\sigma^2}))}{g'(R_2)\sqrt{\hat{V}}}.$$

For the raw statistic, $T_{g,h} = T_h$. By the delta method, it follows from (2) that $T_{g,h} \xrightarrow{d} N(0,1)$.

3. THE BOOTSTRAP

Under stochastic volatility, intraday returns are independent but heteroskedastic, conditional on the volatility path, which motivates a WB in this context. The i.i.d. bootstrap is motivated by a benchmark model in which $\mu_t = 0$ and $\sigma_t = \sigma > 0$ for all t. In this case, intraday returns at horizon h are i.i.d. $N(0, \sigma^2 h)$. As we show here, the i.i.d. bootstrap remains asymptotically valid for general stochastic volatility models described by (1) and (3).

We denote the bootstrap intraday h-period returns as r_i^*. For the i.i.d. bootstrap, r_i^* is i.i.d. from $\{r_i : i = 1, \dots, 1/h\}$. For the WB, $r_i^* = r_i\eta_i$, where η_i are i.i.d. with moments given by $\mu_q^* = E^*|\eta_i|^q$. In the following, P^* denotes the probability measure induced by the bootstrap, conditional on the original sample. Similarly, we let E^* (and Var^*) denote expectation (and variance) with respect to the bootstrap data, conditional on the original sample.

The bootstrap realized volatility is equal to $R_2^* = \sum_{i=1}^{1/h} r_i^{*2}$. For the i.i.d. bootstrap, we can show that $E^*(R_2^*) = R_2$ and $V^* \equiv \mathrm{Var}^*(\sqrt{h^{-1}}R_2^*) = R_4 - R_2^2$.

We propose the following consistent estimator of the i.i.d. bootstrap variance V^*:

$$\hat{V}^* = h^{-1}\sum_{i=1}^{1/h} r_i^{*4} - \left(\sum_{i=1}^{1/h} r_i^{*2}\right)^2 \equiv R_4^* - R_2^{*2}, \tag{4}$$

where for any $q > 0$ we let $R_q^* \equiv h^{-q/2+1}\sum_{i=1}^{1/h} |r_i^*|^q$. The i.i.d. bootstrap analogue of $T_{g,h}$ is given by

$$T_{g,h}^* \equiv \frac{\sqrt{h^{-1}}(g(R_2^*) - g(R_2))}{g'(R_2^*)\sqrt{\hat{V}^*}}. \tag{5}$$

Note that although we center the (transformed) bootstrap realized volatility around the (transformed) sample realized volatility (since $E^*(R_2^*) = R_2$), the bootstrap standard error estimator is not of the same form as that used to studentize $T_{g,h}$. In particular, $\hat{V}^*$ is not given by $\frac{2}{3}R_4^*$, which would be the bootstrap analogue of $\hat{V}$. The naive estimator $\frac{2}{3}R_4^*$ is not consistent for V^* because it relies on a local Gaussianity assumption that does not hold for the i.i.d. nonparametric bootstrap. In contrast, $\hat{V}^*$ given in (4) is a consistent estimator of V^*.

For the WB, we can show that $E^*(R_2^*) = \mu_2^* R_2$ and $V^* \equiv \text{Var}^*(\sqrt{h^{-1}}R_2^*) = (\mu_4^* - \mu_2^{*2})R_4$. We propose the following consistent estimator of V^*,

$$\hat{V}^* = \left(\frac{\mu_4^* - \mu_2^{*2}}{\mu_4^*}\right)R_4^*, \tag{6}$$

and define the WB studentized statistic $T_{g,h}^*$ as

$$T_{g,h}^* \equiv \frac{\sqrt{h^{-1}}(g(R_2^*) - g(\mu_2^* R_2))}{g'(R_2^*)\sqrt{\hat{V}^*}}. \tag{7}$$

Note that $T_{g,h}^*$ is invariant to multiplication of η by a constant when $g(z) = z$ and when $g(z) = \log(z)$, the two leading choices of g.

THEOREM 3.1: *Suppose* (1) *and* (3) *hold. Let $T_{g,h}^*$ denote either the i.i.d. bootstrap statistic defined in* (4) *and* (5), *or the WB statistic defined in* (6) *and* (7). *For the WB, let $\eta_i \sim$ i.i.d. such that $\mu_8^* = E^*|\eta_i|^8 < \infty$. Under Assumption* G, *as $h \to 0$, $\sup_{x\in\mathbb{R}} |P^*(T_{g,h}^* \le x) - P(T_{g,h} \le x)| \xrightarrow{P} 0$.*

This result provides a theoretical justification for using the i.i.d. bootstrap or the WB to consistently estimate the distribution of $T_{g,h}$ for any function g satisfying Assumption G. The conditions under which the i.i.d. bootstrap and WB work are those of BNGJS (2006), which allow for the presence of drifts and leverage effects. As the proof of Theorem 3.1 shows, the asymptotic validity of the bootstrap depends on the availability of a CLT result for R_2 and a law of large numbers for R_q, which hold under the general assumptions of BNGJS (2006).

4. SECOND-ORDER ACCURACY OF THE BOOTSTRAP

We investigate the ability of the bootstrap to provide a second-order asymptotic refinement over the standard normal approximation when estimating $P(T_{g,h} \leq x)$. We make the following assumption.

ASSUMPTION H: *The log price process follows* (1) *with* $\mu_t = 0$ *and* σ_t *is independent of* W_t, *where* σ *is a cadlag process, bounded away from zero, and satisfies* $\lim_{h\to 0} h^{1/2} \sum_{i=1}^{1/h} |\sigma^r_{\eta_i} - \sigma^r_{\xi_i}| = 0$ *for some* $r > 0$, *and for any* η_i *and* ξ_i *such that* $0 \leq \xi_1 \leq \eta_1 \leq h \leq \xi_2 \leq \eta_2 \leq 2h \leq \cdots \leq \xi_{1/h} \leq \eta_{1/h} \leq 1$.

Assumption H restricts considerably our previous assumptions by ruling out drift and leverage effects. The effect of the drift on $T_{g,h}$ is $O_P(\sqrt{h})$ (see, e.g., Meddahi (2002)). While this is asymptotically negligible at the first order, it is not at higher orders. Thus, our higher order results do not allow for $\mu_t \neq 0$. One could in principle bootstrap the centered returns to account for the presence of a constant drift, but we do not explore this possibility here. The no-leverage assumption is mathematically convenient to derive the asymptotic expansions because it allows us to condition on the path of volatility when computing higher order cumulants. Relaxing this assumption is beyond the scope of this paper.

To describe the Edgeworth expansions, we need to introduce some additional notation. We write $\kappa_j(T_{g,h})$ to denote the jth-order cumulant of $T_{g,h}$ and write $\kappa^*_j(T^*_{g,h})$ to denote the corresponding bootstrap cumulant. For $j = 1$ and 3, $\kappa_{j,g}$ denotes the coefficient of the terms of order $O(\sqrt{h})$ of the asymptotic expansion of $\kappa_j(T_{g,h})$, whereas for $j = 2$ and 4, $\kappa_{j,g}$ denotes the coefficients of the terms of order $O(h)$. The bootstrap coefficients $\kappa^*_{j,g,h}$ are defined similarly. For the raw statistic, we omit the subscript g, and write κ_j and $\kappa^*_{j,h}$ to denote the corresponding cumulants. We follow this convention throughout, for instance, when referring to $q_{1,g}(x)$ and $q_{2,g}(x)$. See Appendix A for a precise definition of $\kappa_{j,g}$ and $\kappa^*_{j,g,h}$. Finally, we let $\sigma_{q,p} \equiv \overline{\sigma^q}/(\overline{\sigma^p})^{q/p}$ for any q, $p > 0$. Note that under constant volatility, $\sigma_{q,p} = 1$. Similarly, we let $R_{q,p} = R_q/R_p^{q/p}$.

The formal[3] second-order Edgeworth expansion of the distribution of $T_{g,h}$ can be written as

$$P(T_{g,h} \leq x) = \Phi(x) + \sqrt{h}q_{1,g}(x)\phi(x) + O(h), \tag{8}$$

[3]We do not prove the validity of our Edgeworth expansions. Such a result would be a valuable contribution in itself, which we defer for future research. Here our focus is on using formal expansions to explain the superior finite sample properties of the bootstrap theoretically. See Mammen (1993) and Davidson and Flachaire (2001) for a similar approach.

uniformly over $x \in \mathbb{R}$, where $\Phi(x)$ and $\phi(x)$ are the standard normal cumulative and partial distribution functions, respectively. Following Hall (1992, p. 48), $q_{1,g}(x) = -(\kappa_{1,g} + \frac{1}{6}\kappa_{3,g}(x^2 - 1))$. Given (8), the error of the normal approximation is

$$P(T_{g,h} \leq x) - \Phi(x) = \sqrt{h} q_{1,g}(x)\phi(x) + O(h) \tag{9}$$

uniformly in $x \in \mathbb{R}$. The Edgeworth expansion for the bootstrap is

$$P^*(T^*_{g,h} \leq x) = \Phi(x) + \sqrt{h} q^*_{1,g}(x)\phi(x) + O_P(h), \tag{10}$$

where $q^*_{1,g}(x) = -(\kappa^*_{1,g,h} + \frac{1}{6}\kappa^*_{3,g,h}(x^2 - 1))$.

PROPOSITION 4.1: *Under Assumptions* G *and* H, *conditionally on* σ, *we have that the following statements*:

(a) $q_{1,g}(x) = q_1(x) + \frac{1}{2}(g''(\overline{\sigma^2}))/(g'(\overline{\sigma^2}))\sqrt{2\overline{\sigma^4}}x^2$, *where* $q_1(x) \equiv ((4(2x^2 + 1))/6\sqrt{2})\sigma_{6,4}$.

(b) *For the i.i.d. bootstrap*, $q^*_{1,g}(x) = q^*_1(x) + \frac{1}{2}(g''(R_2))/(g'(R_2))\sqrt{R_4 - R_2^2}x^2$, *where*

$$q^*_1(x) \equiv \frac{1}{6}(2x^2 + 1)\frac{R_6 - 3R_4R_2 + 2R_2^3}{(R_4 - R_2^2)^{3/2}}.$$

(c) *For the WB*, $q^*_{1,g}(x) = q^*_1(x) + \frac{1}{2}(g''(\mu^*_2 R_2))/(g'(\mu^*_2 R_2))\sqrt{(\mu^*_4 - \mu^{*2}_2)R_4}x^2$, *where*

$$q^*_1(x) \equiv -\left(-\frac{A^*_1}{2} + \frac{1}{6}(B^*_1 - 3A^*_1)(x^2 - 1)\right)R_{6,4},$$

$$A^*_1 = \frac{\mu^*_6 - \mu^*_2\mu^*_4}{\mu^*_4(\mu^*_4 - \mu^{*2}_2)^{1/2}},$$

$$B^*_1 = \frac{\mu^*_6 - 3\mu^*_2\mu^*_4 + 2\mu^{*3}_2}{(\mu^*_4 - \mu^{*2}_2)^{3/2}}.$$

Proposition 4.1(a) shows that the magnitude of $q_{1,g}(x)$ depends on σ (except when volatility is constant) and on g. When $g(z) = z$, $q_{1,g}(x) = q_1(x)$ and when $g(z) = \log z$, $q_{1,\log}(x) \equiv q_{1,g}(x) = q_1(x) - \frac{1}{2}\sqrt{2\sigma_{4,2}}x^2$. The following result compares $|q_{1,g}(x)|$ for these two leading choices of g.

PROPOSITION 4.2: *Under Assumption* H, *conditionally on* σ, *for any* $x \neq 0$, $|q_{1,\log}(x)| < |q_1(x)|$ *and* $|q_{1,\log}(0)| = |q_1(0)|$.

Given (9), $\sup_x |q_{1,\log}(x)|/|q_1(x)|$ is a measure of the relative asymptotic error of the normal when approximating the distribution of the log transformed statistic as compared to the raw statistic (to $O(\sqrt{h})$). Proposition 4.2 implies that the error of the normal approximation is larger for the raw statistic than for its log version. This theoretical result explains the finite sample improvements of the log statistic found in the simulations (see BNS (2005) and Section 6).

Gonçalves and Meddahi (2007) applied the results of Proposition 4.1(a) to the class of Box–Cox transforms to show that there are other choices of nonlinear transformations within this class that dominate the log.

Similarly, Gonçalves and Meddahi (2008) use $q_1(x)$ to build improved confidence intervals for $\overline{\sigma^2}$. Although these outperform the CLT-based intervals, they are dominated by the i.i.d. bootstrap intervals proposed here. Recently, Zhang, Mykland, and Aït-Sahalia (2005a) also derived Edgeworth expansions for test statistics based on realized volatility measures. Zhang, Mykland, and Aït-Sahalia (2005a) allowed for microstructure noise (from which we abstract here) and therefore studied a variety of estimators including realized volatility as well as other microstructure noise robust estimators. Nevertheless, their results apply only to normalized statistics based on the true variance of realized volatility (which is unknown in practice), whereas we provide results for the feasible studentized statistics. As Gonçalves and Meddahi (2008) showed, confidence intervals based on Edgeworth expansions for normalized statistics have poor finite sample properties when compared to the Edgeworth-based intervals derived from the correct expansions for the feasible statistics.

For the raw statistic, the second-order Edgeworth expansion for the i.i.d. bootstrap can be obtained as a special case of Liu's (1988) work. She showed that the i.i.d. bootstrap is not only asymptotically valid, but also second-order correct for studentized statistics based on the sample mean of independent but heterogeneous observations. Liu's (1988) results apply to t and bootstrap t statistics that are both studentized by the sample variance. Crucial to Liu's (1988) results is a homogeneity condition on the population means that ensures consistency of the sample variance estimator in the heterogeneous context. Specifically, Liu (1988) assumed that $n^{-1}\sum_{i=1}^{n}(\mu_i - \bar{\mu})^2 \to 0$, where $\mu_i \equiv E(X_i)$, $\bar{\mu} \equiv n^{-1}\sum_{i=1}^{n}\mu_i$, and n is the sample size. Letting $X_i \equiv r_i^2/h$, where $r_i = \sigma_i u_i$, with $u_i \sim N(0,1)$, and letting $n \equiv 1/h$, we can write $R_2 = n^{-1}\sum_{i=1}^{n} X_i$. Conditionally on σ, X_i is independently distributed with mean $\mu_i \equiv \sigma_i^2/h$ and variance $2\sigma_i^4/h^2$. We can show that $q_1^*(x)$ can be obtained from (2.7) in Liu (1988) as a special case. In our context, Liu's (1988) homogeneity condition is $n^{-1}\sum_{i=1}^{n}(\mu_i - \bar{\mu})^2 = \overline{\sigma_h^4} - (\overline{\sigma_h^2})^2 \to 0$, which is not satisfied under stochastic volatility. Thus, we cannot use $R_4 - R_2^2$ to studentize realized volatility. $T_{g,h}$ is the statistic of interest here and this is not covered by the results in Liu (1988). Hence the results in Proposition 4.1(a) are new (and so are the results for the WB, as well as the results for nonlinear functions g for the i.i.d. bootstrap).

Given (10), the bootstrap error in estimating $P(T_{g,h} \leq x)$ is

$$
\begin{aligned}
(11)\qquad & P^*(T^*_{g,h} \leq x) - P(T_{g,h} \leq x) \\
&= \sqrt{h}\Big(\operatorname*{plim}_{h\to 0} q^*_{1,g}(x) - q_{1,g}(x)\Big)\phi(x) + o_P(\sqrt{h})
\end{aligned}
$$

uniformly in $x \in \mathbb{R}$. Next we characterize $\operatorname{plim}_{h\to 0} q^*_{1,g}(x) - q_{1,g}(x)$ for our two bootstrap methods.

4.1. *The i.i.d. Bootstrap Error*

PROPOSITION 4.3: *Under Assumptions G and H, conditionally on σ, we have that the following statements*:

(a) $\operatorname{plim}_{h\to 0} q^*_{1,g}(x) - q_{1,g}(x) = \operatorname{plim}_{h\to 0} q^*_1(x) - q_1(x) + \frac{1}{2}(g''(\overline{\sigma^2}))/(g'(\overline{\sigma^2})) \times (\sqrt{3\overline{\sigma^4} - (\overline{\sigma^2})^2} - \sqrt{2\overline{\sigma^4}})x^2$, *where*

$$
\begin{aligned}
&\operatorname*{plim}_{h\to 0} q^*_1(x) - q_1(x) \\
&\quad = \frac{1}{6}(2x^2+1)\left(\frac{15\overline{\sigma^6} - 9\overline{\sigma^4}\,\overline{\sigma^2} + 2(\overline{\sigma^2})^3}{\big(3\overline{\sigma^4} - (\overline{\sigma^2})^2\big)^{3/2}} - \frac{4}{\sqrt{2}}\frac{\overline{\sigma^6}}{(\overline{\sigma^4})^{3/2}}\right).
\end{aligned}
$$

(b) *If $\sigma_t = \sigma$ for all t, then* $\operatorname{plim}_{h\to 0} q^*_{1,g}(x) - q_{1,g}(x) = 0$.

(c) $|\operatorname{plim}_{h\to 0} q^*_1(x) - q_1(x)| \leq |q_1(x)|$ *uniformly in x*.

Proposition 4.3(a) shows that under Assumptions G and H, $\operatorname{plim}_{h\to 0} q^*_{1,g}(x) - q_{1,g}(x) \neq 0$, implying that the bootstrap error is of the same order, $O_P(\sqrt{h})$, as the normal approximation error. The i.i.d. bootstrap does not match the cumulants of the original statistic when volatility is time-varying, explaining the lack of asymptotic refinements (although it is asymptotically valid, as we showed in Section 3 under more general assumptions than Assumption H). When volatility is constant, Proposition 4.3(b) implies that the i.i.d. bootstrap error is $o_P(\sqrt{h})$, smaller than the normal error $O(\sqrt{h})$. In this case, r_i is i.i.d. $N(0, h\sigma^2)$ and the i.i.d. bootstrap provides a second-order refinement. This result holds for any choice of g, including the raw statistic and the log-based statistic.

When the two approximations have the same convergence rate, an alternative bootstrap accuracy measure is the relative asymptotic error of the bootstrap. See Shao and Tu (1995, Section 3.3) and Davidson and Flachaire (2001) for more on alternative measures of accuracy of the bootstrap. The asymptotic relative bootstrap error can be approximated to $O(\sqrt{h})$ by the ratio $r_{1,g}(x) = |\operatorname{plim}_{h\to 0} q^*_{1,g}(x) - q_{1,g}(x)|/|q_{1,g}(x)|$ for any $x \in \mathbb{R}$. An approximation to this order of the relative error for i.i.d. bootstrap critical values is $r_{1,g}(z_\alpha)$, where z_α is such that $\Phi(z_\alpha) = \alpha$.

For the raw statistic, Proposition 4.3(c) proves that $r_{1,g}(x) \equiv r_1(x) \leq 1$ uniformly in x. Thus, $r_1(z_\alpha) \leq 1$, showing that the bootstrap critical values are more accurate than the normal critical values for the raw statistic under our assumptions. In this case, it is easy to see that $r_1(x)$ is a random function that depends on σ, but not on x. This not only simplifies the proof that $\sup_{x\in\mathbb{R}} r_1(x) \leq 1$, but also allows us to evaluate easily by simulation the magnitude of this ratio for different stochastic volatility models. In particular, we show that this ratio is very small and close to zero for the generalized autoregression conditional heteroskedasticity GARCH(1, 1) diffusion (with a mean of 0.0025 and a maximum of 0.024 across 10,000 simulations), and slightly larger for the two-factor diffusion model (the mean is 0.089 and the maximum is 0.219). See Section 6 for details on the simulation design.

For nonlinear functions g, $r_{1,g}(x)$ is a more complicated function, depending on both σ and x. Proving that $\sup_{x\in\mathbb{R}} r_{1,g}(x) \leq 1$ is therefore more challenging. Although we do not provide a proof of this analytical result, we evaluated by simulation the value of $r_{1,g}(x)$ on a grid of values of x in the interval $[0, 10]$ for $g(z) = \log z$. For the GARCH(1, 1) model, the maximum (over x) mean value (over σ) of $r_{1,\log}(x)$ was 0.0074, with an overall maximum (over σ and x) equal to 0.043. For the two-factor model, these numbers were 0.097 and 0.289 respectively. We take this as evidence of the superior accuracy of the bootstrap critical values for the GARCH(1, 1) and two-factor diffusions, consistent with the good performance of the i.i.d. bootstrap for these models for one-sided intervals based on the log transform (see Section 6).

4.2. *The Wild Bootstrap Error*

PROPOSITION 4.4: *Under Assumptions G and H, conditionally on σ,*

$$\begin{aligned}
&\operatorname*{plim}_{h\to 0} q^*_{1,g}(x) - q_{1,g}(x) \\
&\quad = -\left[\left(\operatorname*{plim}_{h\to 0} \kappa^*_{1,g,h} - \kappa_{1,g}\right) + \frac{1}{6}\left(\operatorname*{plim}_{h\to 0} \kappa^*_{3,g,h} - \kappa_{3,g}\right)(x^2 - 1)\right],
\end{aligned}$$

where

$$\begin{aligned}
\operatorname*{plim}_{h\to 0} \kappa^*_{1,g,h} - \kappa_{1,g} &= -\frac{1}{2}\sigma_{6,4}\left(\frac{5}{\sqrt{3}}A^*_1 - \frac{4}{\sqrt{2}}\right) \\
&\quad - \frac{1}{2}\left(\frac{g''(\mu^*_2\overline{\sigma^2})}{g'(\mu^*_2\overline{\sigma^2})}\sqrt{3\overline{\sigma^4}(\mu^*_4 - \mu^{*2}_2)} - \frac{g''(\overline{\sigma^2})}{g'(\overline{\sigma^2})}\sqrt{2\overline{\sigma^4}}\right), \\
\operatorname*{plim}_{h\to 0} \kappa^*_{3,g,h} - \kappa_{3,g} &= 6\left(\operatorname*{plim}_{h\to 0} \kappa^*_{1,g,h} - \kappa_{1,g}\right) + \sigma_{6,4}\left(\frac{5}{\sqrt{3}}B^*_1 - \frac{4}{\sqrt{2}}\right),
\end{aligned}$$

with A_1^* and B_1^* as in Proposition 4.1.

Proposition 4.4 shows that the ability of the WB to match $\kappa_{1,g}$ and $\kappa_{3,g}$ (and hence provide a second-order asymptotic refinement) depends on g, A_1^*, and B_1^*. The constants A_1^* and B_1^* are a function of μ_q^* for $q = 2, 4, 6$, and therefore depend on the choice of η_i. For instance, if we choose[4] $\eta_i \sim N(0, 1)$, then $A_1^* = A_1 = B_1 = B_1^*$. This implies that for the raw statistic $\operatorname{plim}_{h\to 0} \kappa_{1,h}^* - \kappa_1 = (\frac{5}{\sqrt{3}} - 1)\kappa_1 \neq 0$, and $\operatorname{plim}_{h\to 0} \kappa_{3,h}^* - \kappa_3 = (\frac{5}{\sqrt{3}} - 1)\kappa_3 \neq 0$. In this case, $\operatorname{plim}_{h\to 0} q_1^*(x) - q_1(x) \approx 1.89 q_1(x)$, showing that this choice of η_i does not deliver an asymptotic refinement. It also shows that the contribution of the term $O(\sqrt{h})$ to the bootstrap error is almost twice as large as the contribution of $q_1(x)$ to the normal error. Thus $\eta_i \sim N(0, 1)$ is not a good choice for the WB, which is confirmed by our simulations in Section 6.

A sufficient condition for the WB to provide a second-order asymptotic refinement is that μ_2^*, μ_4^*, and μ_6^* solve $\operatorname{plim}_{h\to 0} \kappa_{1,g,h}^* = \kappa_{1,g}$ and $\operatorname{plim}_{h\to 0} \kappa_{3,g,h}^* = \kappa_{3,g}$. For the raw statistic, as Proposition 4.4 shows, this is equivalent to solving $\frac{5}{\sqrt{3}} A_1^* = \frac{4}{\sqrt{2}}$ and $\frac{5}{\sqrt{3}} B_1^* = \frac{4}{\sqrt{2}}$. We can show that for any $\gamma \neq 0$, the solution is of the form $\mu_2^* = \gamma^2$, $\mu_4^* = \frac{31}{25}\gamma^4$, and $\mu_6^* = \frac{31}{25}\frac{37}{25}\gamma^6$. Since T_h^* is invariant to the choice of γ, we choose $\gamma = 1$ without loss of generality, implying $\mu_2^* = 1$, $\mu_4^* = \frac{31}{25} = 1.24$, and $\mu_6^* = \frac{31}{25}\frac{37}{25} = 1.8352$. Next, we propose a two-point distribution for η_i that matches these three moments and thus implies a second-order asymptotic refinement for the WB for the raw statistic.

PROPOSITION 4.5: *Let T_h^* be defined as in (6) and (7) with $g(z) = z$, and let η_i be i.i.d. such that*

$$\eta_i = \begin{cases} \dfrac{1}{5}\sqrt{31 + \sqrt{186}} \approx 1.33 & \text{with prob } p = \dfrac{1}{2} - \dfrac{3}{\sqrt{186}} \approx 0.28, \\ -\dfrac{1}{5}\sqrt{31 - \sqrt{186}} \approx -0.83 & \text{with prob } 1 - p. \end{cases}$$

Under Assumption H, conditionally on σ, as $h \to 0$, $\sup_{x\in\mathbb{R}} |P^*(T_h^* \leq x) - P(T_h \leq x)| = o_P(\sqrt{h})$.

The choice of η_i in Proposition 4.5 is not optimal for other choices of g, including the log statistic. In this case, the solution to $\operatorname{plim}_{h\to 0} \kappa_{1,g,h}^* = \kappa_{1,g}$ and $\operatorname{plim}_{h\to 0} \kappa_{3,g,h}^* = \kappa_{3,g}$ depends on g and on the volatility path through $\overline{\sigma^q}$. Although we could replace these unknowns by consistent estimates, the Edgeworth expansions derived here would likely change because they do not take into account the randomness of the estimates. In addition, these estimates are

[4]Given that returns are (conditionally on σ) normally distributed, choosing $\eta_i \sim N(0, 1)$ could be a natural choice.

very noisy and it is unclear whether such an approach would be useful in practice. See Gonçalves and Meddahi (2007) for more on a related issue. For these reasons, we do not pursue this approach here.

5. THIRD-ORDER ACCURACY OF THE BOOTSTRAP

Here we develop Edgeworth expansions through $O(h)$ and use these to evaluate the accuracy of the bootstrap for estimating $P(|T_h| \leq x)$. For brevity, we only give results for the raw statistic. The third-order Edgeworth expansion of the distribution of T_h is

$$P(T_h \leq x) = \Phi(x) + \sqrt{h} q_1(x)\phi(x) + h q_2(x)\phi(x) + o(h) \tag{12}$$

for any $x \in \mathbb{R}$, where q_1 is defined in Section 4 and q_2 is an odd polynomial of degree 5 whose coefficients depend on κ_j for $j = 1, \ldots, 4$. The third-order bootstrap Edgeworth expansion is similar to (12), with $q_1^*(x)$ and $q_2^*(x)$ denoting the bootstrap analogues of $q_1(x)$ and $q_2(x)$, respectively. In particular, $q_2^*(x)$ is of the same form as $q_2(x)$ but replaces the coefficients κ_j with bootstrap analogues $\kappa_{j,h}^*$.

The error in estimating $P(|T_h| \leq x)$ made by the normal approximation is given by $P(|T_h| \leq x) - (2\Phi(x) - 1) = 2hq_2(x)\phi(x) + o(h)$, which is $O(h)$. The bootstrap error can be written as

$$P^*(|T_h^*| \leq x) - P(|T_h| \leq x) = 2h\Big[\operatorname*{plim}_{h\to 0} q_2^*(x) - q_2(x)\Big]\phi(x) + o_P(h). \tag{13}$$

The bootstrap provides a third-order asymptotic refinement when $\text{plim}_{h\to 0}\, q_2^*(x) = q_2(x)$ or, equivalently, when $\text{plim}_{h\to 0}\, \kappa_{j,h}^* = \kappa_j$ for $j = 1, \ldots, 4$.

Our findings are as follows. The i.i.d. bootstrap does not provide third-order asymptotic refinements. This is true even when volatility is constant, which is a surprising result. Under constant volatility, $\text{plim}_{h\to 0}\, \kappa_{j,h}^* = \kappa_j$ for $j = 1$ and 3 (implying that $\text{plim}_{h\to 0}\, q_1^*(x) = q_1(x)$; cf. Proposition 4.3(b)), but this is not true for $j = 2$ and 4. Note that this does not mean that the i.i.d. bootstrap provides inconsistent estimates of the asymptotic value (as $h \to 0$) of the second and fourth cumulants of T_h. Since $\kappa_2^*(T_h^*) = 1 + h\kappa_{2,h}^* + o_P(h)$ and $\kappa_4^*(T_h^*) = h\kappa_{4,h}^* + o_P(h)$, it follows that $\text{plim}_{h\to 0}\, \kappa_2^*(T_h^*) = 1 = \text{plim}_{h\to 0}\, \kappa_2(T_h)$ and $\text{plim}_{h\to 0}\, \kappa_4^*(T_h^*) = 0 = \text{plim}_{h\to 0}\, \kappa_4(T_h)$, independently of the value of $\text{plim}_{h\to 0}\, \kappa_{j,h}^*$ and κ_j; these terms are multiplied by h, which goes to zero, and only play a role in proving bootstrap refinements.

The reason why the i.i.d. bootstrap does not provide a third-order asymptotic refinement under constant volatility is related to the fact that T_h^* uses a variance estimator $\hat{V}^*$ which is not the bootstrap analogue of the variance estimator $\hat{V} \equiv \frac{2}{3}R_4$ used in T_h. Under constant volatility, an alternative consistent variance estimator of the asymptotic variance of R_2 is $\tilde{V} = R_4 - R_2^2$, which is of the same form as $\hat{V}^*$. We can show that for a t statistic based on $\tilde{V}$, we get second- and third-order asymptotic refinements for the i.i.d. bootstrap under constant volatility. Using $\hat{V}$ instead of $\tilde{V}$ does not have an impact at the second order, but it does at the third order. Because $\tilde{V}$ is only consistent for V under constant volatility, we cannot use it in the general context of stochastic volatility.

Our main finding for the WB is that there is no choice of η_i for which the WB gives a third-order asymptotic refinement. In particular, it is not possible to find η_i such that $\text{plim}_{h\to 0}\kappa^*_{j,h} = \kappa_j$ for $j = 1, \dots, 4$. As discussed in Section 4, to match the first- and third-order cumulants, we need to choose η_i with moments $\mu^*_2 = \gamma^2$, $\mu^*_4 = \frac{31}{25}\gamma^4$, and $\mu^*_6 = \frac{31}{25}\frac{37}{25}\gamma^6$. Since the WB statistic is invariant to the choice of γ, we set $\gamma = 1$. We are left with two equations ($\text{plim}_{h\to 0}\kappa^*_{j,h} = \kappa_j$ for $j = 2, 4$) and one free parameter μ^*_8. The two-point distribution proposed in Proposition 4.5 gives a second-order refinement, implying $\mu^*_8 = 3.014$. We can also choose η_i to solve $\text{plim}_{h\to 0}\kappa^*_{j,h} = \kappa_j$ for $j = 1, 2, 3$ by setting $\mu^*_2 = 1$, $\mu^*_4 = \frac{31}{25}$, $\mu^*_6 = \frac{31}{25}\frac{37}{25}$, and $\mu^*_8 = (\frac{31}{25})^2(\frac{1}{25})(\frac{1739}{35}) = 3.056$.[5] Because it solves $\text{plim}_{h\to 0}\kappa^*_{j,h} = \kappa_j$ for $j = 2$ (in addition to $j = 1$ and 3), this choice may perform better than the two-point choice of η_i in Proposition 4.5.

Given the absence of third-order bootstrap asymptotic refinements, we rely on the asymptotic relative error of the bootstrap as the criterion of comparison. To $O(h)$, this error is equal to $r_2(x) = |\text{plim}_{h\to 0} q^*_2(x) - q_2(x)|/|q_2(x)|$, with $x > 0$. In the general stochastic volatility case, $r_2(x)$ is a random function of x as it depends on σ through the ratios $\sigma_{6,4}$ and $\sigma_{8,4}$. When σ is constant, these ratios equal 1 and $r_2(x)$ becomes a deterministic function of x. Figure 1 plots $r_2(x)$ against x when σ is constant. Four methods are considered: the i.i.d. bootstrap, the WB based on $\eta_i \sim N(0, 1)$, the WB based on η_i chosen according to Proposition 4.5, and a third WB whose moments μ^*_q solve $\text{plim}_{h\to 0}\kappa^*_{j,h} = \kappa_j$ for $j = 1, 2, 3$. Figure 1 shows that $\sup_x r_2(x) < 1$ for the i.i.d. bootstrap, suggesting that it is better than the normal approximation under this criterion. Instead, Figure 1 shows that for the WB, $r_2(x)$ can be larger or smaller than 1, depending on x, except for the WB based on $N(0, 1)$, for which it is always well above 1. We also evaluated $r_2(x)$ by simulation when σ is stochastic, as we did for $r_{1,\log}(x)$. The results show that $r_2(x)$ can be smaller or larger than 1, depending on x. Overall, Figure 1 suggests that the asymptotic relative bootstrap error criterion is not a good indicator of the accuracy of our WB methods for two-sided distribution functions. Although Edgeworth expansions are the main theoretical tool for proving bootstrap asymptotic refinements, it has already been pointed out in the bootstrap literature (see, e.g., Härdle, Horowitz, and Kreiss (2003)) that Edgeworth expansions can be imperfect guides to the relative accuracy of the bootstrap methods. The same comment applies here to the asymptotic relative bootstrap error criterion for two-sided distribution functions.

[5]Matching κ_j for $j = 1, 2, 4$ is not possible because the solution for the μ^*_q's does not satisfy Jensen's inequality.

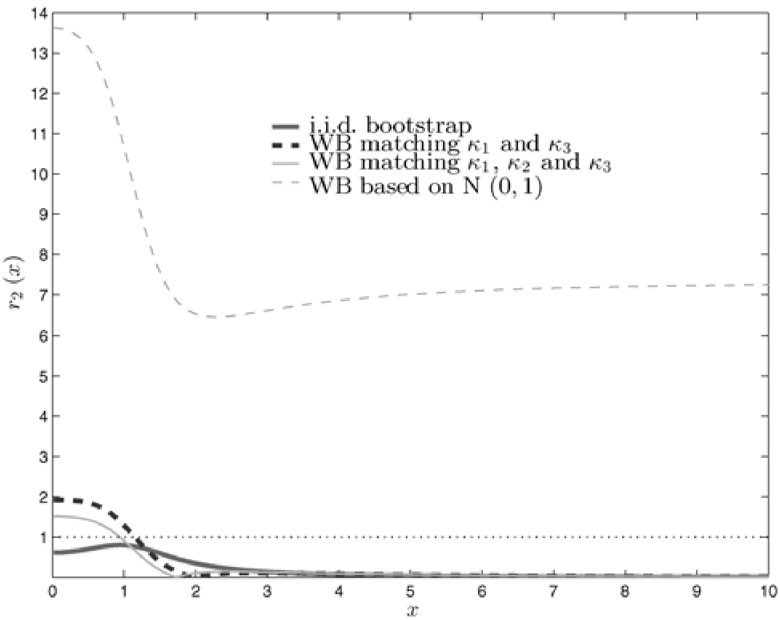

FIGURE 1.—The function $r_2(x)$ when σ is constant.

6. MONTE CARLO RESULTS

We compare the finite sample performance of the bootstrap with the first-order asymptotic theory for confidence intervals of integrated volatility. Our Monte Carlo design follows that of Andersen, Bollerslev, and Meddahi (2005). In particular, we consider the stochastic volatility model

$$d\log S_t = \mu\, dt + \sigma_t\big[\rho_1\, dW_{1t} + \rho_2\, dW_{2t} + \sqrt{1-\rho_1^2-\rho_2^2}\, dW_{3t}\big],$$

where W_{1t}, W_{2t}, and W_{3t} are three independent standard Brownian motions. For σ_t, we consider a GARCH(1, 1) diffusion (cf. Andersen and Bollerslev (1998)), where $d\sigma_t^2 = 0.035(0.636-\sigma_t^2)\, dt + 0.144\sigma_t^2\, dW_{1t}$, and a two-factor diffusion (see Huang and Tauchen (2006) and Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008)) where $\sigma_t = \exp(-1.2 + 0.04\sigma_{1t}^2 + 1.5\sigma_{2t}^2)$, with $d\sigma_{1t}^2 = -0.00137\sigma_{1t}^2\, dt + dW_{1t}$ and $d\sigma_{2t}^2 = -1.386\sigma_{2t}^2\, dt + (1+0.25\sigma_{2t}^2)\, dW_{2t}$.

Our baseline models let $\mu = 0$ and $\rho_1 = \rho_2 = 0$, consistent with Assumption H. We also allow for drift and leverage effects by setting $\mu = 0.0314$, $\rho_1 = -0.576$, and $\rho_2 = 0$ for the GARCH(1, 1) model, and $\mu = 0.030$ and $\rho_1 = \rho_2 = -0.30$ for the two-factor diffusion model, for which our results in Section 3 apply. We consider one- and two-sided symmetric 95% confidence intervals based on the raw and on the log statistics. We use the normal distribution (CLT), the i.i.d. bootstrap (iidB), and two WB methods, one based on

$\eta_i \sim N(0,1)$ (WB1) and another based on the two-point distribution proposed in Proposition 4.5 (WB2) to compute critical values.

Table I gives the actual coverage rates of all the intervals across 10,000 replications for four different sample sizes: $1/h = 1152$, 288, 48, and 12, corresponding to 1.25-minute, 5-minute, half-hour, and 2-hour returns. Bootstrap intervals use 999 bootstrap replications. For all models, both one-sided and two-sided asymptotic intervals tend to undercover. The degree of undercoverage is especially large for larger values of h, when sampling is not too frequent, and it is larger for one-sided than for two-sided intervals. It is also larger for the raw statistics than for the log-based statistics. The two-factor model implies overall lower coverage rates (hence larger coverage distortions) than the GARCH(1, 1) model. The bootstrap methods outperform the feasible asymptotic theory for both one- and two-sided intervals, and for the raw and the log statistics. The i.i.d. bootstrap does very well across all models and intervals, even though there is stochastic volatility. It essentially eliminates the distortions associated with the asymptotic intervals for small values of $1/h$ for the GARCH(1, 1). Its performance deteriorates for the two-factor model, but it remains very competitive relative to the other methods. The WB intervals based on the normal distribution tend to overcover across all models. The WB based on the two-point distribution tends to undercover, but significantly less than the feasible asymptotic theory intervals. This is true for both the raw and the log versions of R_2, although its relative performance is worse for the log case, for which this choice is not optimal. The i.i.d. and the WB based on the two-point distribution outperform the normal approximation for symmetric intervals, despite the fact that these bootstrap methods do not theoretically provide an asymptotic refinement for two-sided symmetric confidence intervals. The i.i.d. bootstrap is the preferred method overall, followed by the WB based on the proposed two-point distribution. Finally, the results are robust to leverage and drift effects.

7. CONCLUSIONS

The results presented here justify using the i.i.d. bootstrap and the wild bootstrap for a class of nonlinear transformations of realized volatility that contains the log transform as a special case. We show that these methods are asymptotically valid under the assumptions of BNGJS (2006), which allow for drift and leverage effects. In simulations, the bootstrap is more accurate than the standard normal asymptotic theory for two popular stochastic volatility models. We provide higher order results that explain these findings under a stricter set of assumptions that rules out drift and leverage effects. Establishing higher order refinements of the bootstrap under the conditions of BNGJS (2006) is a promising extension of this work. Another important extension is to prove the validity of the Edgeworth expansions derived here. Finally, one interesting application of the bootstrap is to realized beta, where the Monte Carlo results of

TABLE I

COVERAGE RATES OF NOMINAL 95% CONFIDENCE INTERVALS FOR $\overline{\sigma^2}$ [a]

	One-Sided								Two-Sided Symmetric							
	Raw				Log				Raw				Log			
$1/h$	CLT	iidB	WB1	WB2	CLT	iidB	WB1	WB2	CLT	iidB	WB1	WB2	CLT	iidB	WB1	WB2
	Baseline Models: No Leverage and No Drift															
	GARCH(1, 1) diffusion															
12	82.69	93.27	98.51	87.50	88.83	93.48	98.07	90.27	86.08	93.75	98.51	87.49	90.40	95.86	97.96	88.30
48	89.74	94.63	98.32	93.87	92.74	94.74	97.73	95.20	92.32	94.87	98.32	93.83	93.64	95.46	97.42	94.66
288	93.03	95.10	97.40	95.04	94.33	95.12	97.03	95.55	94.57	95.18	97.05	95.17	94.70	95.11	96.38	95.13
1152	94.01	95.02	96.51	95.04	94.56	95.00	96.22	95.21	94.81	94.97	95.69	94.88	94.85	94.99	95.43	94.86
	Two-factor diffusion															
12	75.69	89.70	96.52	78.94	82.41	90.35	96.12	82.76	78.94	90.13	96.52	78.92	85.90	93.32	96.14	80.25
48	84.52	92.66	96.92	89.71	88.48	92.64	96.49	91.70	87.95	92.83	96.92	89.79	90.85	93.97	96.50	90.95
288	90.27	94.28	97.32	93.49	92.12	94.25	96.94	94.35	92.83	94.59	97.25	93.98	93.59	94.88	96.78	94.27
1152	93.20	95.02	96.93	94.95	94.04	94.99	96.60	95.30	94.64	95.20	96.52	94.89	94.77	95.11	96.08	94.92
	Models With Leverage and Drift															
	GARCH(1, 1) diffusion															
12	82.40	93.00	98.36	87.21	88.40	93.32	98.04	89.99	85.72	93.69	98.36	87.22	90.48	95.70	97.93	88.29
48	89.81	94.70	98.57	94.01	92.72	94.79	98.01	95.17	92.35	94.97	98.57	93.92	93.65	95.55	97.70	94.66
288	92.84	94.98	97.37	94.95	94.25	95.00	96.87	95.46	94.41	95.15	96.84	94.94	94.56	95.09	96.19	94.80
1152	94.28	95.16	96.70	95.13	94.77	95.16	96.27	95.39	95.04	95.13	96.05	95.13	95.10	95.16	95.59	95.15
	Two-factor diffusion															
12	75.79	90.44	96.75	79.57	83.09	90.67	96.34	82.97	79.52	90.87	96.75	79.55	86.09	93.50	96.34	80.40
48	84.16	92.69	97.05	89.68	88.51	92.76	96.60	91.73	87.81	92.89	97.05	89.69	90.76	94.08	96.57	90.82
288	90.75	94.56	97.34	93.76	92.39	94.57	97.04	94.69	93.14	94.81	97.30	94.08	93.76	94.99	96.68	94.36
1152	93.01	95.13	96.79	94.82	93.98	95.08	96.54	95.17	94.27	94.81	96.33	94.56	94.47	94.88	95.84	94.74

[a]Notes: CLT—intervals based on the Normal; iidB—intervals based on the i.i.d. bootstrap; WB1—WB based on $\eta_i \sim N(0, 1)$; WB2—WB based on Proposition 4.5. 10,000 Monte Carlo trials with 999 bootstrap replications each.

BNS (2004a) show that there are important finite sample distortions. Dovonon, Gonçalves, and Meddahi (2007) considered this extension.

APPENDIX A: CUMULANT EXPANSIONS

This Appendix contains the cumulant expansions used in the paper. Auxiliary lemmas and proofs appear in the Supplemental Material (see GM09). Recall that $\sigma_{q,p} \equiv \overline{\sigma^q}/(\overline{\sigma^p})^{q/p}$ for any $q, p > 0$. In some results, $\overline{\sigma^q}$ is replaced with $\overline{\sigma_h^q}$ in this definition and we write $\sigma_{q,p,h}$. Finally, we let $R_{q,p} \equiv R_q/(R_p)^{q/p}$.

THEOREM A.1—Cumulants of $T_{g,h}$: *Suppose Assumptions* G *and* H *hold*. *For any* $q > 0$, $\overline{\sigma_h^q} - \overline{\sigma^q} = o_P(\sqrt{h})$, *and*, *conditionally on* σ, *as* $h \to 0$,

(a) $\kappa_1(T_h) = \sqrt{h}\kappa_1 + o(h)$, *with* $\kappa_1 \equiv -(A_1/2)\sigma_{6,4}$;

(b) $\kappa_1(T_{g,h}) = \sqrt{h}\kappa_{1,g} + O(h)$, *with* $\kappa_{1,g} \equiv \kappa_1 - \frac{1}{2}(g''(\overline{\sigma^2}))/(g'(\overline{\sigma^2}))\sqrt{2\overline{\sigma^4}}$;

(c) $\kappa_2(T_h) = 1 + h\kappa_2 + o(h)$, *with* $\kappa_2 \equiv (C_1 - A_2)\sigma_{8,4} + \frac{7}{4}A_1^2\sigma_{6,4}^2$;

(d) $\kappa_3(T_h) = \sqrt{h}\kappa_3 + o(h)$, *with* $\kappa_3 \equiv (B_1 - 3A_1)\sigma_{6,4}$;

(e) $\kappa_3(T_{g,h}) = \sqrt{h}\kappa_{3,g} + O(h)$, *with* $\kappa_{3,g} \equiv \kappa_3 - 3(g''(\overline{\sigma^2}))/(g'(\overline{\sigma^2}))\sqrt{2\overline{\sigma^4}}$;

(f) $\kappa_4(T_h) = h\kappa_4 + o(h)$, *with* $\kappa_4 \equiv (B_2 + 3C_1 - 6A_2)\sigma_{8,4} + (18A_1^2 - 6A_1B_1) \times \sigma_{6,4}^2$, *and*

$$A_1 = \frac{\mu_6 - \mu_2\mu_4}{\mu_4(\mu_4 - \mu_2^2)^{1/2}} = \frac{4}{\sqrt{2}},$$

$$A_2 = \frac{\mu_8 - \mu_4^2 - 2\mu_2\mu_6 + 2\mu_2^2\mu_4}{\mu_4(\mu_4 - \mu_2^2)} = 12,$$

$$B_1 = \frac{\mu_6 - 3\mu_2\mu_4 + 2\mu_2^3}{(\mu_4 - \mu_2^2)^{3/2}} = \frac{4}{\sqrt{2}},$$

$$B_2 = \frac{\mu_8 - 4\mu_2\mu_6 + 12\mu_2^2\mu_4 - 6\mu_2^4 - 3\mu_4^2}{(\mu_4 - \mu_2^2)^2} = 12,$$

$$C_1 = \frac{\mu_8 - \mu_4^2}{\mu_4^2} = \frac{32}{3}.$$

THEOREM A.2—i.i.d. Bootstrap Cumulants: *Under Assumptions* G *and* H, *conditionally on* σ, *as* $h \to 0$,

(a) $\kappa_1^*(T_h^*) = \sqrt{h}\kappa_{1,h}^* + o_P(h)$, *with* $\kappa_{1,h}^* \equiv -\tilde{A}_1/2$;

(b) $\kappa_1^*(T_{g,h}^*) = \sqrt{h}\kappa_{1,g,h}^* + O_P(h)$, *with* $\kappa_{1,g,h}^* \equiv \kappa_{1,h}^* - \frac{1}{2}(g''(R_2))/(g'(R_2)) \times \sqrt{R_4 - R_2^2}$;

(c) $\kappa_2^*(T_h^*) = 1 + h\kappa_{2,h}^* + o_P(h)$, *with* $\kappa_{2,h}^* \equiv \tilde{C} - \tilde{A}_2 - \frac{1}{4}\tilde{A}_1^2$;

(d) $\kappa_3^*(T_h^*) = \sqrt{h}\kappa_{3,h}^* + o_P(h)$, *with* $\kappa_{3,h}^* \equiv -2\tilde{A}_1$;

(e) $\kappa_3^*(T_{g,h}^*) = \sqrt{h}\kappa_{3,g,h}^* + O_P(h)$, *with* $\kappa_{3,g,h}^* \equiv \kappa_{3,h}^* - 3(g''(R_2))/(g'(R_2)) \times \sqrt{R_4 - R_2^2}$;

(f) $\kappa_4^*(T_h^*) = h\kappa_{4,h}^* + o_P(h)$, *with* $\kappa_{4,h}^* \equiv (\tilde{B}_2 - 2\tilde{D} + 3\tilde{E}) - 6(\tilde{C} - \tilde{A}_2) - 4\tilde{A}_1^2$, *where*

$$\tilde{A}_1 = \frac{R_6 - 3R_4R_2 + 2R_2^3}{(R_4 - R_2^2)^{3/2}},$$

$$\tilde{A}_2 = \frac{R_8 - 4R_4^2 - 4R_6R_2 + 14R_4R_2^2 - 7R_2^4}{(R_4 - R_2^2)^2},$$

$$\tilde{B}_2 = \frac{R_8 - 4R_6R_2 + 12R_4R_2^2 - 6R_2^4 - 3R_4^2}{(R_4 - R_2^2)^2},$$

$$\tilde{C} = \frac{R_8 - R_4^2}{(R_4 - R_2^2)^2} + \frac{2(R_6 - R_4R_2)^2}{(R_4 - R_2^2)^3} - \frac{12(R_6 - R_4R_2)(R_2)}{(R_4 - R_2^2)^2} + \frac{12R_2^2}{R_4 - R_2^2},$$

$$\begin{aligned}\tilde{D} = {} & \frac{4(R_6 - 3R_4R_2 + 2R_2^3)(R_6 - R_4R_2)}{(R_4 - R_2^2)^3} \\ & + \frac{6(R_8 - R_4^2 - 2R_6R_2 + 2R_4R_2^2)}{(R_4 - R_2^2)^2} \\ & - 15 - \frac{20R_2(R_6 - 3R_4R_2 + 2R_2^3)}{(R_4 - R_2^2)^2},\end{aligned}$$

$$\begin{aligned}\tilde{E} = {} & \frac{3(R_8 - R_4^2)}{(R_4 - R_2^2)^2} + \frac{12(R_6 - R_4R_2)^2}{(R_4 - R_2^2)^3} \\ & - \frac{60(R_6 - R_4R_2)(R_2)}{(R_4 - R_2^2)^2} + \frac{60(R_2)^2}{R_4 - R_2^2}.\end{aligned}$$

THEOREM A.3—WB Cumulants: *Under Assumptions* G *and* H, *conditionally on* σ, *as* $h \to 0$,

(a) $\kappa_1^*(T_h^*) = \sqrt{h}\kappa_{1,h}^* + o_P(h)$, *with* $\kappa_{1,h}^* \equiv -(A_1^*/2)R_{6,4}$;

(b) $\kappa_1^*(T_{g,h}^*) = \sqrt{h}\kappa_{1,g,h}^* + O_P(h)$, *with* $\kappa_{1,g,h}^* \equiv \kappa_{1,h}^* - \frac{1}{2}(g''(\mu_2^*R_2))/(g'(\mu_2^* \times R_2))\sqrt{(\mu_4^* - \mu_2^{*2})R_4}$;

(c) $\kappa_2^*(T_h^*) = 1 + h\kappa_{2,h}^* + o_P(h)$, *with* $\kappa_{2,h}^* \equiv (C_1^* - A_2^*)R_{8,4} + \frac{7}{4}A_1^{*2}R_{6,4}^2$;

(d) $\kappa_3^*(T_h^*) = \sqrt{h}\kappa_{3,h}^* + o_P(h)$, *with* $\kappa_{3,h}^* \equiv (B_1^* - 3A_1^*)R_{6,4}$;

(e) $\kappa_3^*(T_{g,h}^*) = \sqrt{h}\kappa_{3,g,h}^* + O_P(h)$, *with* $\kappa_{3,g,h}^* \equiv \kappa_{3,h}^* - 3(g''(\mu_2^*R_2))/(g'(\mu_2^* \times R_2))\sqrt{(\mu_4^* - \mu_2^{*2})R_4}$;

(f) $\kappa_4^*(T_h^*) = h\kappa_{4,h}^* + o_P(h)$, *with* $\kappa_{4,h}^* \equiv (B_2^* + 3C_1^* - 6A_2^*)R_{8,4} + (18A_1^{*2} - 6A_1^*B_1^*)R_{6,4}^2$, *where*

$$A_1^* = \frac{\mu_6^* - \mu_2^*\mu_4^*}{\mu_4^*(\mu_4^* - \mu_2^{*2})^{1/2}},$$

$$A_2^* = \frac{\mu_8^* - \mu_4^{*2} - 2\mu_2^*\mu_6^* + 2\mu_2^{*2}\mu_4^*}{\mu_4^*(\mu_4^* - \mu_2^{*2})},$$

$$B_1^* = \frac{\mu_6^* - 3\mu_2^*\mu_4^* + 2\mu_2^{*3}}{(\mu_4^* - \mu_2^{*2})^{3/2}},$$

$$B_2^* = \frac{\mu_8^* - 4\mu_2^*\mu_6^* + 12\mu_2^{*2}\mu_4^* - 6\mu_2^{*4} - 3\mu_4^{*2}}{(\mu_4^* - \mu_2^{*2})^2},$$

$$C_1^* = \frac{\mu_8^* - \mu_4^{*2}}{\mu_4^{*2}}.$$

PROOF OF THEOREM A.1: We sketch the proofs for the raw statistic. The proofs of (b) and (e) for nonlinear g follow by a second-order Taylor expansion of $K(R_2, \hat{V})$ around $(\overline{\sigma^2}, V_h)$, where $K(x, y) = (g(x) - g(\overline{\sigma^2}))/(g'(x)\sqrt{y})$ and $g(\cdot)$ is as in Assumption G. We let $V_h = \mathrm{Var}(\sqrt{h^{-1}}R_2) = 2\overline{\sigma_h^4}$, and let $S_h \equiv (\sqrt{h^{-1}}(R_2 - \overline{\sigma^2}))/\sqrt{V_h}$ and $U_h \equiv (\sqrt{h^{-1}}(\hat{V} - V_h))/V_h$. We can write $T_h = S_h(1 + \sqrt{h}U_h)^{-1/2}$. The first four cumulants of T_h are given by (e.g., Hall (1992, p. 42))

$$\kappa_1(T_h) = E(T_h); \quad \kappa_2(T_h) = E(T_h^2) - [E(T_h)]^2,$$
$$\kappa_3(T_h) = E(T_h^3) - 3E(T_h^2)E(T_h) + 2[E(T_h)]^3,$$
$$\kappa_4(T_h) = E(T_h^4) - 4E(T_h^3)E(T_h) - 3[E(T_h^2)]^2 + 12E(T_h^2)[E(T_h)]^2 - 6[E(T_h)]^4.$$

We identify the terms of order up to $O(h)$. For a fixed k, we can write

$$T_h^k = S_h^k(1 + \sqrt{h}U_h)^{-k/2} = S_h^k - \frac{k}{2}\sqrt{h}S_h^kU_h + \frac{k}{4}\left(\frac{k}{2} + 1\right)hS_h^kU_h^2 + O\big(h^{3/2}\big).$$

For $k = 1, \dots, 4$, the moments of T_h^k up to $O(h^{3/2})$ are given by

$$E(T_h) = -\sqrt{h}\frac{1}{2}E(S_hU_h) + \frac{3}{8}hE(S_hU_h^2),$$

$$E(T_h^2) = 1 - \sqrt{h}E(S_h^2U_h) + hE(S_h^2U_h^2),$$

$$E(T_h^3) = E(S_h^3) - \sqrt{h}\frac{3}{2}E(S_h^3 U_h) + \frac{15}{8}hE(S_h^3 U_h^2),$$
$$E(T_h^4) = E(S_h^4) - 2\sqrt{h}E(S_h^4 U_h) + 3hE(S_h^4 U_h^2),$$

where we used $E(S_h) = 0$ and $E(S_h^2) = 1$. By Lemma S.3 in GM09, we have that

$$E(T_h) = \sqrt{h}\left(-\frac{1}{2}A_1\sigma_{6,4,h}\right) + O\big(h^{3/2}\big),$$
$$E(T_h^2) = 1 + h\big[(C_1 - A_2)\sigma_{8,4,h} + C_2\sigma_{6,4,h}^2\big] + O(h^2),$$
$$E(T_h^3) = \sqrt{h}\left[\left(B_1 - \frac{3}{2}A_3\right)\sigma_{6,4,h}\right] + O\big(h^{3/2}\big),$$
$$E(T_h^4) = 3 + h\big((B_2 - 2D_1 + 3E_1)\sigma_{8,4,h} + (3E_2 - 2D_2)\sigma_{6,4,h}^2\big) + O(h^2).$$

Thus $\kappa_1(T_h) = \sqrt{h}(-(A_1/2)\sigma_{6,4,h}) + O(h^{3/2}) = \sqrt{h}(-(A_1/2)\sigma_{6,4}) + O(h^{3/2})$, since under Assumption H, BNS (2004b) showed that $\overline{\sigma_h^q} - \overline{\sigma^q} = o(h^{1/2})$. This proves the first result. The remaining results follow similarly. *Q.E.D.*

PROOF OF THEOREM A.2: We follow the proof of Theorem A.1 and use Lemma S.7 in GM09 instead of Lemma S.3. The cumulant expansions follow by noting that $\tilde{A}_3 = 3\tilde{A}_1$ and $\tilde{B}_1 = \tilde{A}_1$. *Q.E.D.*

See the proof of Theorem A.1 and Remark 1 in GM09 for the proof of Theorem A.3.

APPENDIX B: PROOFS OF RESULTS IN SECTIONS 3–5

PROOF OF THEOREM 3.1: Given that $T_{g,h} \xrightarrow{d} N(0,1)$, it suffices that $T_{g,h}^* \xrightarrow{d^*} N(0,1)$ in probability. We prove this for $g(z) = z$; the delta method implies the result for nonlinear g. The proof contains two steps: 1. show the desired result for $S_h^* \equiv \sqrt{h^{-1}}(R_2^* - E^*(R_2^*))/\sqrt{V^*}$; 2. show $\hat{V}^* \xrightarrow{P^*} V^*$ in probability.

We start with the i.i.d. bootstrap.

Step 1. Let $S_h^* = \sum_{i=1}^{1/h} z_i^*$, where $z_i^* \equiv (r_i^{*2} - E^*(r_i^{*2}))/\sqrt{hV^*}$ are (conditionally) i.i.d. with $E^*(z_i^*) = 0$ and $\operatorname{Var}^*(z_i^*) = h^2V^*/hV^* = h$ such that $\operatorname{Var}^*(\sum_{i=1}^{1/h} z_i^*) = 1$. Thus, by the Berry–Esseen bound, for some small $\varepsilon > 0$ and some constant K,

$$\sup_{x\in\mathbb{R}}\left|P^*\left(\sum_{i=1}^{1/h} z_i^* \le x\right) - \Phi(x)\right| \le K\sum_{i=1}^{1/h} E^*|z_i^*|^{2+\varepsilon},$$

which converges to zero in probability as $h \to 0$. We have

$$\begin{aligned}\sum_{i=1}^{1/h} E^*|z_i^*|^{2+\varepsilon} &= h^{-1-(2+\varepsilon)/2}|V^*|^{-(2+\varepsilon)/2}E^*\big(|r_1^{*2} - E^*|r_1^*|^2|^{2+\varepsilon}\big)\\ &\le 2|V^*|^{-(2+\varepsilon)/2}h^{-1-(2+\varepsilon)/2}E^*|r_1^*|^{2(2+\varepsilon)}\\ &= 2|V^*|^{-(2+\varepsilon)/2}h^{\varepsilon/2}R_{2(2+\varepsilon)} = O_P\big(h^{\varepsilon/2}\big) = o_P(1),\end{aligned}$$

since $V^* \xrightarrow{P} 3\overline{\sigma^4} - (\overline{\sigma^2})^2 > 0$ and $R_{2(2+\varepsilon)} \xrightarrow{P} \mu_{2(2+\varepsilon)}\overline{\sigma^{2(2+\varepsilon)}} = O(1)$.

Step 2. Use Lemma S.5 in GM09 to show that $\mathrm{Bias}^*(\hat{V}^*) \xrightarrow{P} 0$ and $\mathrm{Var}^*(\hat{V}^*) \xrightarrow{P} 0$. The proof for the WB follows similarly. *Q.E.D.*

PROOF OF PROPOSITION 4.1: The results follow from the definition of $q_{1,g}(x)$ and $q_{1,g}^*(x)$ given the cumulants expansions in Theorems A.1, A.2, and A.3. *Q.E.D.*

The proof of Proposition 4.2 appears in GM09.

PROOF OF PROPOSITION 4.3: (a) We compute $\mathrm{plim}_{h\to 0}\,\kappa_{j,g,h}^*$ for $j = 1, 3$ using Theorem A.2 and the fact that $R_q \xrightarrow{P} \mu_q\overline{\sigma^q}$, as shown by BNGJS (2006). (b) Follows trivially when σ is constant because $(\overline{\sigma^q})^p = \sigma^{qp}$ for any $q, p > 0$. The proof of (c) appears in GM09. *Q.E.D.*

PROOF OF PROPOSITION 4.4: This follows from Theorem A.1 and A.3, given that $R_q \to \mu_q\overline{\sigma^q}$ in probability for any $q > 0$, by BNGJS (2006). *Q.E.D.*

PROOF OF PROPOSITION 4.5: Let $\eta_i = a_1$ with probability p and let $\eta_i = a_2$ with probability $1 - p$. We can show that $a_1 = \frac{1}{5}\sqrt{31 + \sqrt{186}}$, $a_2 = -\frac{1}{5}\sqrt{31 - \sqrt{186}}$, and $p = \frac{1}{2} - \frac{3}{\sqrt{186}}$ solve $E(\eta_i^2) = a_1^2 p + a_2^2(1-p) = 1$, $E(\eta_i^4) = a_1^4 p + a_2^4(1-p) = \frac{31}{25}$, and $E(\eta_i^6) = a_1^6 p + a_2^6(1-p) = \frac{31}{25}\frac{37}{25}$. *Q.E.D.*

REFERENCES

ANDERSEN, T. G., AND T. BOLLERSLEV (1998): "Answering the Skeptics: Yes, Standard Volatility Models Do Provide Accurate Forecasts," *International Economic Review*, 39, 885–905. [297]

ANDERSEN, T. G., T. BOLLERSLEV, AND F. X. DIEBOLD (2002): "Parametric and Nonparametric Measurements of Volatility," in *Handbook of Financial Econometrics*, ed. by Y. Aït-Sahalia and L. P. Hansen. Amsterdam: North Holland (forthcoming). [283]

ANDERSEN, T. G., T. BOLLERSLEV, F. X. DIEBOLD, AND P. LABYS (2003): "Modeling and Forecasting Realized Volatility," *Econometrica*, 71, 529–626. [284]

ANDERSEN, T. G., T. BOLLERSLEV, AND N. MEDDAHI (2005): "Correcting the Errors: Volatility Forecast Evaluation Using High-Frequency Data and Realized Volatilities," *Econometrica*, 73, 279–296. [297]

BANDI, F. M., AND J. R. RUSSELL (2008): "Microstructure Noise, Realized Variance, and Optimal Sampling," *Review of Economic Studies*, 75, 339–369. [284]

BARNDORFF-NIELSEN, O., AND N. SHEPHARD (2002): "Econometric Analysis of Realized Volatility and Its Use in Estimating Stochastic Volatility Models," *Journal of the Royal Statistical Society, Ser. B*, 64, 253–280. [283,285,287]

——— (2004a): "Econometric Analysis of Realized Covariation: High Frequency Based Covariance, Regression, and Correlation in Financial Economics," *Econometrica*, 72, 885–925. [300]

——— (2004b): "Power and Bipower Variation With Stochastic Volatility and Jumps," *Journal of Financial Econometrics*, 2, 1–48. [303]

——— (2005): "How Accurate Is the Asymptotic Approximation to the Distribution of Realised Volatility?" in *Inference for Econometric Models. A Festschrift for Tom Rothenberg*, ed. by D. W. K. Andrews and J. H. Stock. Cambridge, U.K.: Cambridge University Press, 306–331. [284,291]

BARNDORFF-NIELSEN, O., S. E. GRAVERSEN, J. JACOD, AND N. SHEPHARD (2006): "Limit Theorems for Bipower Variation in Financial Econometrics," *Econometric Theory*, 22, 677–719. [283,286-289,298,304]

BARNDORFF-NIELSEN, O., P. R. HANSEN, A. LUNDE, AND N. SHEPHARD (2008): "Designing Realized Kernels to Measure the ex post Variation of Equity Prices in the Presence of Noise," *Econometrica*, 76, 1481–1536. [284,297]

DAVIDSON, R., AND E. FLACHAIRE (2001): "The Wild Bootstrap, Tamed at Last," Working Paper Darp58, STICERD, London School of Economics. [285,289,292]

DOVONON, P., S. GONÇALVES, AND N. MEDDAHI (2007): "Bootstrapping Realized Multivariate Volatility Measures," Manuscript, Université de Montréal. [300]

GONÇALVES, S., AND N. MEDDAHI (2007): "Box–Cox Transforms for Realized Volatility," Manuscript, Université de Montréal. [291,295]

——— (2008): "Edgeworth Corrections for Realized Volatility," *Econometric Reviews*, 27, 139–162. [291]

——— (2009): "Supplement to 'Bootstrapping Realized Volatility'," *Econometrica Supplemental Material*, 77, http://www.econometricsociety.org/ecta/Supmat/5971_Miscellaneous.pdf. [286]

HALL, P. (1992): *The Bootstrap and Edgeworth Expansion*. New York: Springer Verlag. [290,302]

HANSEN, P. R., AND A. LUNDE (2006): "Realized Variance and Market Microstructure" (With Discussion), *Journal of Business and Economic Statistics*, 24, 127–218. [284]

HÄRDLE, W., J. HOROWITZ, AND J.-P. KREISS (2003): "Bootstrap Methods for Time Series," *International Statistical Review*, 71, 435–459. [296]

HUANG, X., AND G. TAUCHEN (2006): "The Relative Contribution of Jumps to Total Price Variance," *Journal of Financial Econometrics*, 3, 456–499. [297]

JACOD, J., AND P. PROTTER (1998): "Asymptotic Error Distributions for the Euler Method for Stochastic Differential Equations," *Annals of Probability*, 26, 267–307. [287]

LIU, R. Y. (1988): "Bootstrap Procedure Under Some Non-i.i.d. Models," *Annals of Statistics*, 16, 1696–1708. [284,291]

MAMMEN, E. (1993): "Bootstrap and Wild Bootstrap for High Dimensional Linear Models," *Annals of Statistics*, 21, 255–285. [284,289]

MEDDAHI, N. (2002): "A Theoretical Comparison Between Integrated and Realized Volatility," *Journal of Applied Econometrics*, 17, 475–508. [289]

SHAO, J., AND D. TU (1995): *The Jackknife and Bootstrap*. New York: Springer Verlag. [285,292]

WU, C. F. J. (1986): "Jackknife, Bootstrap and Other Resampling Methods in Regression Analysis," *Annals of Statistics*, 14, 1261–1295. [284]

ZHANG, L., P. MYKLAND, AND Y. AÏT-SAHALIA (2005a): "Edgeworth Expansions for Realized Volatility and Related Estimators," Working Paper, Princeton University. [291]

——— (2005b): "A Tale of Two Time Scales: Determining Integrated Volatility With Noisy High-Frequency Data," *Journal of the American Statistical Association*, 100, 1394–1411. [284]

Département de Sciences Économiques, CIREQ and CIRANO, Université de Montréal, C.P. 6128, Succ. Centre-Ville, Montréal, QC, H3C 3J7, Canada; silvia.goncalves@umontreal.ca

and

Finance and Accounting Group, Tanaka Business School, Imperial College London, Exhibition Road, London SW7 2AZ, U.K.; n.meddahi@imperial.ac.uk.

Manuscript received July, 2005; final revision received June, 2008.